Lecture 6
Methods for Point Estimator

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Methods of Point Estimation

- Method of Moments (MoM)
- Method of Maximum Likelihood
- Bayesian methods
- …

Methods of Moments

Population and samples moments

Let $X_1, X_2, \ldots, X_n$ be a random sample from the probability distribution $f(x)$, where $f(x)$ can be a discrete probability mass function or a continuous probability density function. The kth **population moment** (or **distribution moment**) is $E(X^k)$, $k = 1, 2, \ldots$. The corresponding kth **sample moment** is $(1/n)\sum_{i=1}^{n} X_i^k$, $k = 1, 2, \ldots$.

Population moments

$$\mu_k' = \begin{cases} \int_x x^k f(x)\,dx & \text{If } x \text{ is continuous} \\ \sum_x x^k f(x) & \text{If } x \text{ is discrete} \end{cases}$$

Sample moments

$$m_k' = \frac{\sum_{i=1}^{n} X_i^k}{n}$$

Methods of Moments

Let $X_1, X_2, \ldots, X_n$ be a random sample from either a probability mass function or probability density function with m unknown parameters $\theta_1, \theta_2, \ldots, \theta_m$. The **moment estimators** $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_m$ are found by equating the first m population moments to the first m sample moments and solving the resulting equations for the unknown parameters.

m equations for *m* parameters

$$\begin{cases} m_1' = \mu_1' \\ m_2' = \mu_2' \\ \vdots \\ m_m' = \mu_m' \end{cases}$$

Example

1) What is the point estimator of λ in the exponential distribution?

2) What is the point estimator of p in the Binomial distribution?

3) What is the point estimator for mean and variance in normal distribution?

Methods of Point Estimation

- Method of Moments (MoM)
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Method of Maximum Likelihood

Suppose that X is a random variable with probability distribution $f(x; \theta)$, where θ is a single unknown parameter. Let $x_1, x_2, \ldots, x_n$ be the observed values in a random sample of size n. Then the **likelihood function** of the sample is

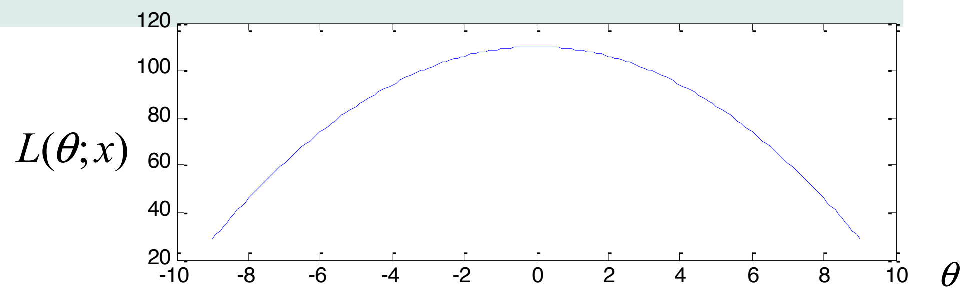
$$L(\theta) = f(x_1; \theta) \cdot f(x_2; \theta) \cdot \cdots \cdot f(x_n; \theta) \qquad (7\text{-}9)$$

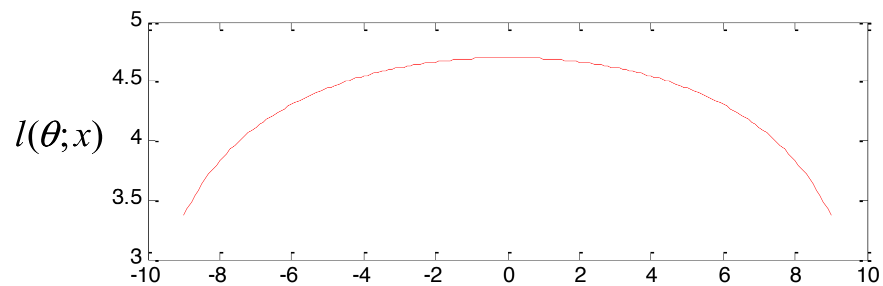
Note that the likelihood function is now a function of only the unknown parameter θ. The **maximum likelihood estimator (MLE)** of θ is the value of θ that maximizes the likelihood function $L(\theta)$.

$$L(\theta;x) = \prod_{i=1}^{n} f(x_i;\theta) = f(x_1;\theta)\ldots f(x_n;\theta)$$

$$l(\theta;x) = \sum_{i=}^{n} \log\left[f(x_i;\theta)\right]$$

$$\hat{\Theta}(x) = \arg\max_{\theta} L(\theta;x) = \arg\max_{\theta} l(\theta;x)$$





Example

Let X be a Bernoulli random variable. The probability mass function is

$$f(x;p) = \begin{cases} p^x(1-p)^{1-x}, & x = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

where p is the parameter to be estimated. The likelihood function of a random sample of size n is

$$\begin{aligned} L(p) &= p^{x_1}(1-p)^{1-x_1} p^{x_2}(1-p)^{1-x_2} \cdots p^{x_n}(1-p)^{1-x_n} \\ &= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i} \end{aligned}$$

$$\ln L(p) = \left(\sum_{i=1}^{n} x_i\right) \ln p + \left(n - \sum_{i=1}^{n} x_i\right) \ln(1-p)$$

$$\frac{d \ln L(p)}{dp} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\left(n - \sum_{i=1}^{n} x_i\right)}{1-p} \longrightarrow \hat{P} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Example

Let X be normally distributed with mean μ and variance σ^2, where both μ and σ^2 are unknown. The likelihood function for a random sample of size n is

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x_i-\mu)^2/(2\sigma^2)} = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-(1/2\sigma^2)\sum_{i=1}^{n}(x_i-\mu)^2}$$

and

$$\ln L(\mu, \sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Example (Continued)

$$\frac{\partial \ln L(\mu, \sigma^2)}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) = 0$$

$$\frac{\partial \ln L(\mu, \sigma^2)}{\partial (\sigma^2)} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i - \mu)^2 = 0$$

The solutions to the above equation yield the maximum likelihood estimators

$$\hat{\mu} = \overline{X} \qquad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

Once again, the maximum likelihood estimators are equal to the moment estimators.

Exponential MLE

Let X be a exponential random variable with parameter λ. The likelihood function of a random sample of size n is:

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\ln L(\lambda) = n \ln(\lambda) - \lambda \sum_{i=1}^{n} x_i$$

$$\frac{d \ln L(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\hat{\lambda} = n \Big/ \sum_{i=1}^{n} x_i = 1/\bar{X} \quad \text{(same as moment estimator)}$$

MLE Properties

Under very general and not restrictive conditions, when the sample size n is large and if $\hat{\Theta}$ is the maximum likelihood estimator of the parameter θ,

(1) $\hat{\Theta}$ is an approximately unbiased estimator for θ $[E(\hat{\Theta}) \simeq \theta]$,

(2) the variance of $\hat{\Theta}$ is nearly as small as the variance that could be obtained with any other estimator, and

(3) $\hat{\Theta}$ has an approximate normal distribution.

Example:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

Invariance Property

Let $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_k$ be the maximum likelihood estimators of the parameters $\theta_1, \theta_2, \ldots, \theta_k$. Then the maximum likelihood estimator of any function $h(\theta_1, \theta_2, \ldots, \theta_k)$ of these parameters is the same function $h(\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_k)$ of the estimators $\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_k$.

Example:

In the normal distribution case, the maximum likelihood estimators of μ and σ^2 were $\hat{\mu} = \overline{X}$ and $\hat{\sigma}^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2/n$. To obtain the maximum likelihood estimator of the function $h(\mu, \sigma^2) = \sqrt{\sigma^2} = \sigma$, substitute the estimators $\hat{\mu}$ and $\hat{\sigma}^2$ into the function h, which yields

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2\right]^{1/2}$$

Thus, the maximum likelihood estimator of the standard deviation σ is *not* the sample standard deviation S.

Complications in Using MLE

- It is not always easy to maximize the likelihood function because the equation(s) obtained from setting derivative to be 0 may be difficult to solve.
- It may not always be possible to use calculus methods directly to determine the maximum of the likelihood function.

Example: Uniform Distribution MLE

Let *X* be uniformly distributed on the interval 0 to a.

$$f(x) = 1/a \text{ for } 0 \le x \le a$$

$$L(a) = \prod_{i=1}^{n} \frac{1}{a} = \frac{1}{a^n} = a^{-n} \text{ for } 0 \le x_i \le a$$

$$\frac{dL(a)}{da} = \frac{-n}{a^{n+1}} = -na^{-(n+1)}$$

$$\hat{a} = \max(x_i)$$

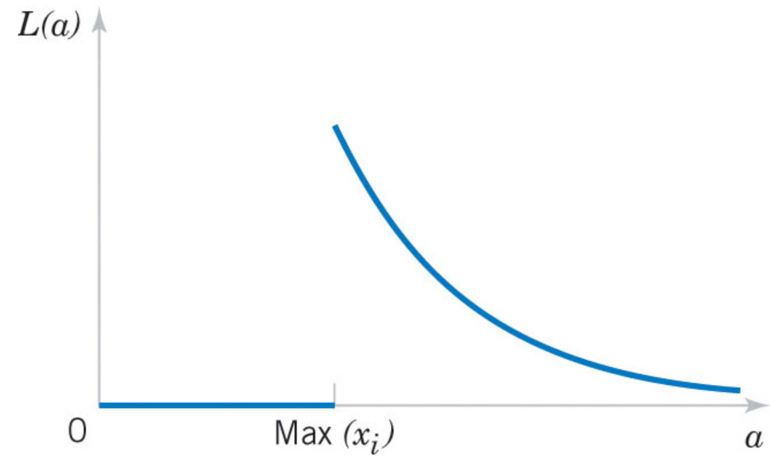


Figure 7-8 The likelihood function for this uniform distribution

Calculus methods don't work here because *L*(*a*) is maximized at the discontinuity.
Clearly, *a* cannot be smaller than max(x_i), thus the MLE is max(x_i).