assume $p \le 1/2$

(P1) For Hamming distortion

$$d(x,\hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x} \\ 1 & \text{if } x \ne \hat{x} \end{cases}$$

$$E\,d(x,\hat{x}) = \sum_{(x,\hat{x})} p(x)\,p(\hat{x}|x)\,d(x,\hat{x})$$

$$= \sum_{x \ne \hat{x}} p(x,\hat{x})$$

$$= p(x \ne \hat{x})$$

$$\Rightarrow E\,d(x,\hat{x}) \le D \Rightarrow p(x \ne \hat{x}) \le D$$

Now let's exam $X: \sim \text{Bernolli}(p)$

$$I(x;\hat{x}) = H(X) - H(x|\hat{x})$$

$$= H(p) - H(x \oplus \hat{x} | \hat{x})$$

modulus 2 addition flips sign of x, if we know $\hat{x}$, we can recover x

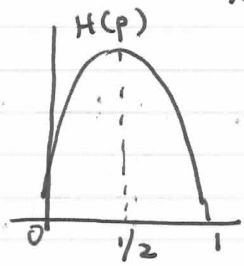
$$\ge H(p) - H(x \oplus \hat{x})$$

$$= H(p) - H(p(x \ne \hat{x}))$$

$$x \oplus \hat{x} = \begin{cases} 1 & x \ne \hat{x} \\ 0 & x = \hat{x} \end{cases}$$

(i)

for $D \le 1/2$
$H(p)$ is increasing



$P(x \ne \hat{x}) \le D$

$$\ge H(p) - H(D) \quad (\text{achieved when } P(x \ne \hat{x}) = D$$

$$\Rightarrow R(D) \ge H(p) - H(D)$$

- this lower-bound is tight, we can find a conditional dist. to meet this bound

① choose $P(x \ne \hat{x}) = \cancel{H(D)}\ D$

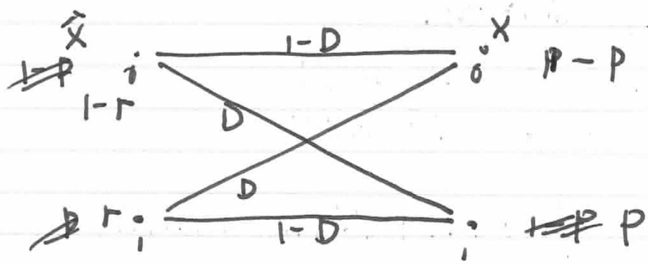
② ~~$H(x \oplus \hat{x} | \hat{x}) = H(x \oplus \hat{x}) = H(D)$~~

~~if $\hat{x} = \begin{cases} 1 & \text{w.p. } r \\ 0 & \text{w.p. } 1-r \end{cases}$~~ ~~$x \oplus \hat{x} = \begin{cases} 1 & x \ne \hat{x} \text{ w.p. } D \\ 0 & \text{w.p. } 1-D \end{cases}$~~

$H(x \oplus \hat{x} | \hat{x}) = H(x \oplus \hat{x})$, channel must be symmetric

P2

③



we only need to find r, s.t.

the ~~outp~~ "x" has desirable distribution

$$r(1-D) + (1-r)D = p$$

$$r - rD + D - Dr = p$$

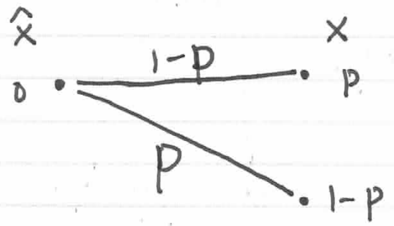
$$(1-2D)r = p - D$$

$$\boxed{r = \frac{p-D}{1-2D}}$$

$I(X;\hat{X}) = H(X) - H(D)$

~~(ii) if $D \le p \le 1/2$~~

(ii) if $D \ge p$, we can achieve $R(D) = 0$ by letting $\hat{X} = 0$ w.p. 1 $(I(X;\hat{X}) = 0)$



set distortion $= p$

(D is the largest distortion)

Finally by symmetry

$$R(D) = \begin{cases} H(p) - H(D), & 0 \le D \le \min\{p, 1-p\} \\ 0 & D > \min\{p, 1-p\} \end{cases}$$

(P3) Gaussian distortion function:

$$X \sim \mathcal{N}(0, \sigma^2)$$

$$R(D) = \min_{f(\hat{x}|x) \,:\, E(\hat{X}-X)^2 \le D} I(X;\hat{X})$$

$$
\begin{aligned}
I(X;\hat{X}) &= h(X) - h(X|\hat{X}) \\
&= \tfrac{1}{2}\log 2\pi e\sigma^2 - h(X-\hat{X}|\hat{X}) \quad \text{(similar argument)} \\
&\ge \tfrac{1}{2}\log 2\pi e\sigma^2 - h(X-\hat{X}) \\
&\ge \tfrac{1}{2}\log(2\pi e\sigma^2) - h(\mathcal{N}(0, E(X-\hat{X})^2)) \\
&= \tfrac{1}{2}\log(2\pi e\sigma^2) - \tfrac{1}{2}\log 2\pi e\, E(X-\hat{X})^2
\end{aligned}
$$

$$E(\hat{X}-X)^2 \le D$$

$$
\begin{aligned}
&\ge \tfrac{1}{2}\log(2\pi e\sigma^2) - \tfrac{1}{2}\log 2\pi e D \\
&= \tfrac{1}{2}\log\frac{\sigma^2}{D}
\end{aligned}
$$

to find $f(\hat{x}|\hat{x})$ that achieves this lower bound, consider the conditional density

$f(x|\hat{x})$: "test channel"

b/c $h(X-\hat{X}|\hat{X}) = h(X-\hat{X})$: must have this kind of channel

$$\hat{X} \longrightarrow \overset{\overset{Z \sim N(0,D)}{\downarrow}}{\oplus} \longrightarrow X \sim N(0, \sigma^2)$$

$\hat{X} \sim N(0, \sigma^2 - D)$

this is by our choice

$$
\begin{aligned}
I(X;\hat{X}) &= \tfrac{1}{2}\log\left[1 + \frac{\sigma^2 - D + D}{D}\right] \\
&= \tfrac{1}{2}\log\frac{\sigma^2}{D}
\end{aligned}
$$

Eg 3

$X \sim$ unif. $(1, 2, \dots m)$

Hamming distortion.

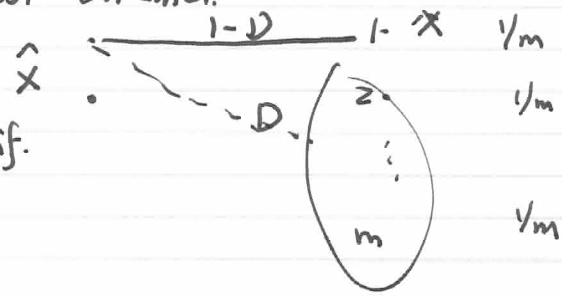
$D \equiv P(X \neq \hat{X})$

Fano: $H(X|\hat{X}) \leq H(D) + \underset{\bar{P}_e}{D}\log(m-1)$

$I(X;\hat{X}) = H(X) - H(X|\hat{X})$

$\geq \log m - H(D) - D\log(m-1)$

test channel.



$\hat{x}$: unif. dist.

$p(x,\hat{x}) = p(x)p(\hat{x}|x)$
$= p(\hat{x})p(x|\hat{x})$

$\cancel{P(\hat{x}|x)}$ $P(x|\hat{x}) = \begin{cases} 1-D & \text{if } \hat{x} = x \\ D/(m-1) & \text{if } \hat{x} \neq x \end{cases}$

$R(D) = \begin{cases} \log m - H(D) - D\log(m-1) & \text{if } 0 \leq D \leq 1 - \frac{1}{m} \\ 0 & D > 1 - \frac{1}{m} \end{cases}$