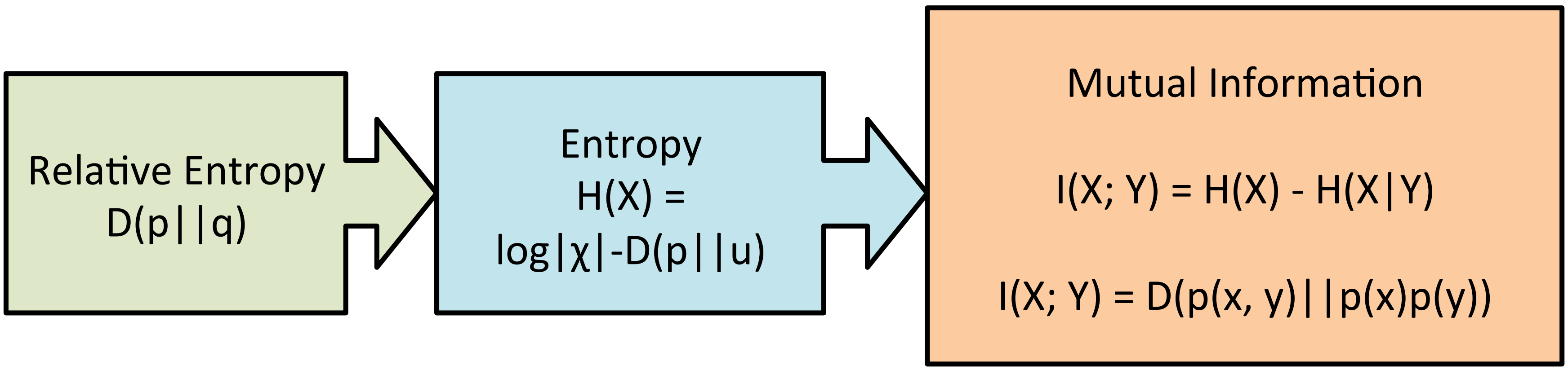
Lecture 3: Chain Rules and Inequalities

- Last lecture: entropy and mutual information
- This time
  - Chain rules
  - Jensen's inequality
  - Log-sum inequality
  - Concavity of entropy
  - Convex/concavity of mutual information

Logic order of things



Chain rule for entropy

- Last time, simple chain rule $H(X,Y) = H(X) + H(Y|X)$
- No matter how we play with chain rule, we get the same answer

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

"entropy of two experiments"

Chain rule for entropy

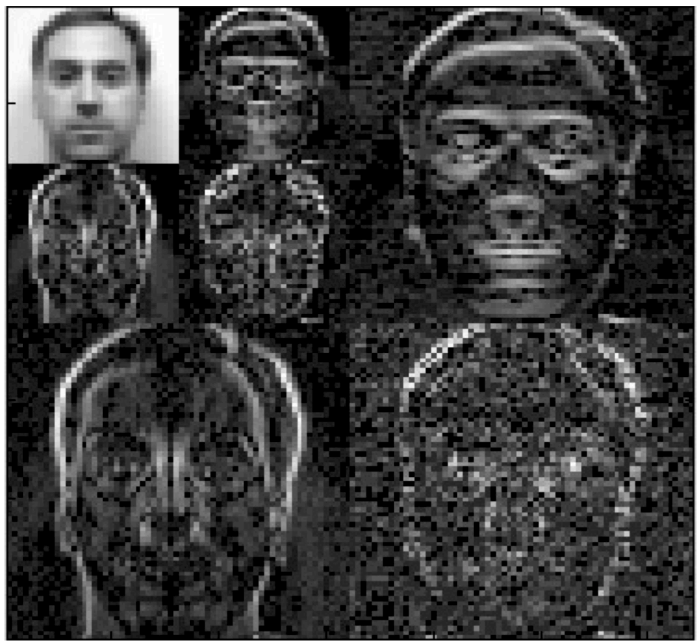
- Entropy for a collection of RV's is the sum of the conditional entropies
- More generally: $H(X_1, X_2, \cdots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \cdots, X_1)$

  Proof:

$$H(X_1, X_2) = H(X_1) + H(X_2|X_1)$$
$$H(X_1, X_2, X_3) = H(X_3, X_2|X_1) + H(X_1)$$
$$= H(X_3|X_2, X_1) + H(X_2|X_1) + H(X_1)$$
$$\vdots$$

Implication on image compression

$$H(X^n) = \sum_{i=1}^{n} H(X_i | \underbrace{X_{-i}}_{\text{everything seen before}})$$



Conditional mutual information

- Definition
$$I(X;Y|Z) = H(X|Z) - H(X|Y,Z)$$
- In our "asking native for weather" example
  - We want to infer X: rainy or sunny
  - Originally, we only know native's answer Y: yes or no. Value of native's answer $I(X;Y)$
  - If we also has a humidity meter with measurement Z. Value of native's answer $I(X;Y|Z)$

Chain rule for mutual information

- Chain rule for information

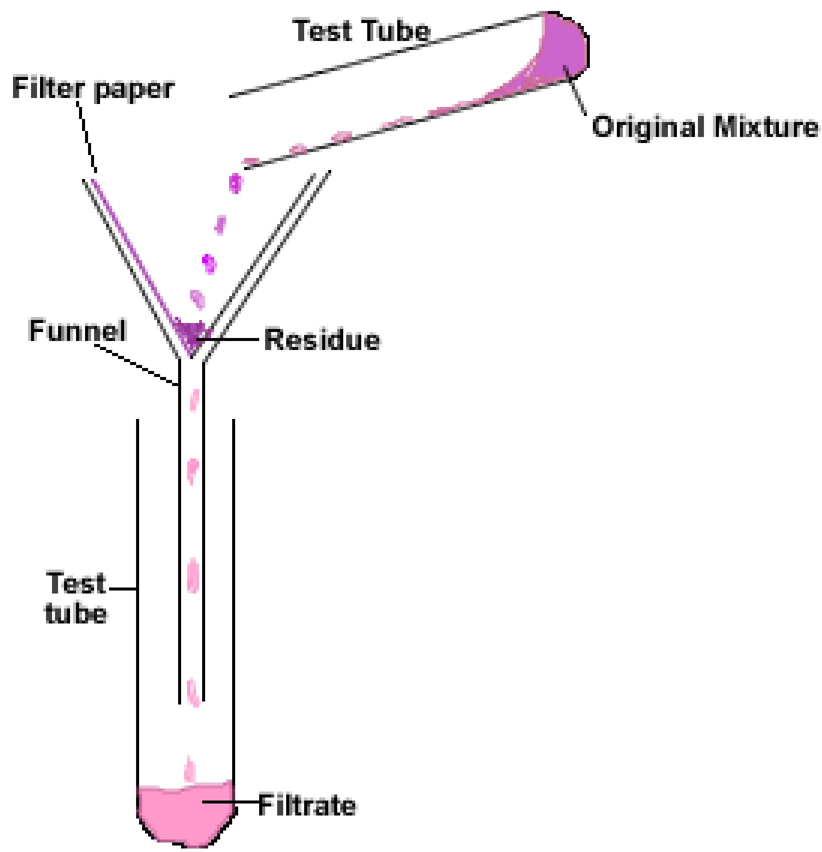
$$I(X_1, X_2, \cdots, X_n; Y) = \sum_{i=1}^{n} I(X_i; Y | X_{i-1}, \cdots, X_1)$$

Proof:

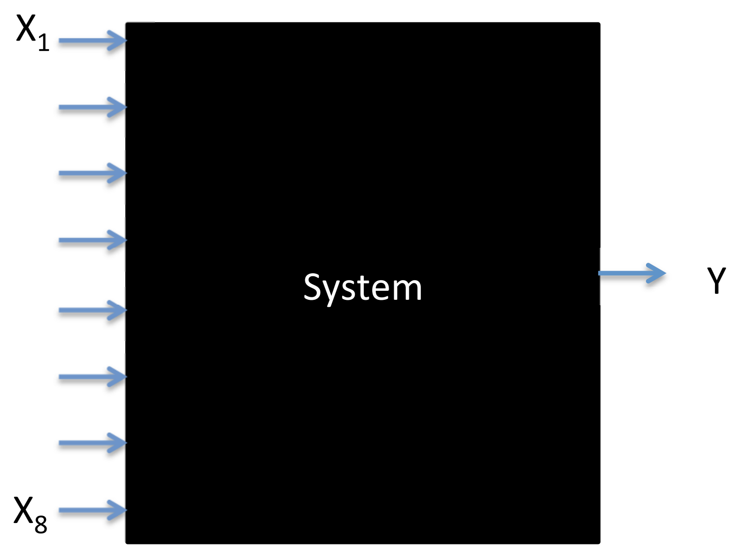
$$I(X_1, X_2, \cdots, X_n; Y) = H(X_1, \cdots, X_n) - H(X_1, \cdots, X_n | Y)$$

Apply chain rules for entropy on both sides.

- Interpretation 1: "Filtration of information"



- Interpretation 2: by observing Y, how many possible inputs $(X_1, \cdots, X_8)$ can be distinguished:
  resolvability of X_i as observed by Y



Conditional relative entropy

- Definition:

$$D(p(y|x)||q(y|x)) = \sum_x p(x) \sum_y p(y|x) \log \frac{p(y|x)}{q(y|x)}$$

- Chain rule for relative entropy

$$D(p(x,y)||q(x,y)) = D(p(x)||q(x)) + D(p(y|x)||q(y|x))$$

Distance between joint pdfs = distances between margins + distance between conditional pdfs

Why do we need inequalities in information theory?

Convexity

- A function $f(x)$ is convex over an interval (a, b) if for every $x, y \in (a, b)$ and $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

  Strictly convex if equality holds only if $\lambda = 0$.

- If a function f has second order derivative $\geq 0 (> 0)$, the function is convex (strictly convex).

- Vector valued function: Hessian matrix is nonnegative definite.

- Examples: x^2, e^x, $|x|$, $x \log x (x \geq 0)$, $\|\boldsymbol{x}\|^2$.

- A function f is concave if $-f$ is convex.

- Linear function $ax + b$ is both convex and concave.

How to show a function is convex

- By definition: $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ (function must be continous)

- Verify $f''(x) \geq 0$ (or nonnegative definite)

- By composition rules:
  - Composition of affine function $f(Ax+b)$ is convex if f is convex
  - Composition with a scalar function: $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$, $f(x) = h(g(x))$, then f is convex if

    (1) g convex, h convex, $\tilde{h}$ nondecreasing

    (2) g concave, h convex, $\tilde{h}$ nonincreasing

    Extended-value extension $\tilde{f}(x) = f(x)$, $x \in \mathcal{X}$, otherwise is ∞

Jensen's inequality

- Due to Danish mathematician Johan Jensen, 1906
- Widely used in mathematics and information theory
- Convex transformation of a mean
  $\leq$ mean after convex transformation

Theorem. *(Jensen's inequality) If f is a convex function,*

$$Ef(X) \geq f(EX).$$

If f strictly convex, equality holds when

$$X = constant.$$

Proof: Let $x^\star = EX$. Expand $f(x)$ by Taylor's Theorem at $x^\star$:

$$f(x) = f(x^\star) + f'(x^\star)(x - x^\star) + \frac{f''(z)}{2}(x - x^\star)^2, \quad z \in (x, x^\star)$$

f convex: $f''(z) \geq 0$. So $f(x) \geq f(x^\star) + f'(x^\star)(x - x^\star)$. Take expectation on both size: $Ef(X) \geq f(x^\star)$.

Consequences

- $f(x) = x^2$, $EX^2 \geq [EX]^2$: variance is nonnegative
- $f(x) = e^x$, $Ee^x \geq e^{E(x)}$
- Arithmetic mean $\geq$ Geometric mean $\geq$ Harmonic mean

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

Proof using Jensen's inequality: $f(x) = x \log x$ is convex.

Information inequality

$$D(p||q) \geq 0,$$

equality iff $p(x) = q(x)$ for all x.
Proof:

$$\begin{aligned}
D(p||q) &= \sum_x p(x) \log \frac{p(x)}{q(x)} \\
&= -\sum_x p(x) \log \frac{q(x)}{p(x)} \\
&\geq \log \sum_x p(x) \frac{q(x)}{p(x)} \\
&= \log \sum_x q(x) = 0.
\end{aligned}$$

- $I(X;Y) \geq 0$, equality iff X and Y are independent.
  Since $I(X;Y) = D(p(x,y)||p(x)p(y))$.

- Conditional relative entropy and mutual information are also nonnegative

Conditioning reduces entropy

Information cannot hurt:

$$H(X|Y) \leq H(X)$$

- Since $I(X;Y) = H(X) - H(X|Y) \geq 0$
- Knowing another RV Y only reduces uncertainty in X on average
- $H(X|Y = y)$ may be larger than $H(X)$: in court, knowing a new evidence sometimes can increase uncertainty

Independence bound on entropy

$$H(X_1, \cdots, X_n) \leq \sum_{n=1}^{n} H(X_i).$$

equality iff X_i independent.

- From chain rule:

$$H(X_1, \cdots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \cdots, X_1) \leq \sum_{i=1}^{n} H(X_i).$$

Maximum entropy

Uniform distribution has maximum entropy among all distributions with finite discrete support.

Theorem. $H(X) \leq \log |\mathcal{X}|$, *where* $\mathcal{X}$ *is the number of elements in the set. Equality iff* X *has uniform distribution.*

Proof: Let U be a uniform distributed RV, $u(x) = 1/|\mathcal{X}|$

$$0 \leq D(p||u) = \sum p(x) \log \frac{p(x)}{u(x)} \tag{1}$$

$$= \sum p(x) \log |\mathcal{X}| - (-\sum p(x) \log p(x)) = \log |\mathcal{X}| - H(X) \tag{2}$$

Log sum inequality

- Consequence of concavity of log

  Theorem. *For nonnegative* $a_1, \cdots, a_n$ *and* $b_1, \cdots, b_n$

  $$\sum_{i=1}^n a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^n a_i\right) \log \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}.$$

  Equality iff $a_i/b_i = $ *constant.*

- Proof by Jensen's inequality using convexity of $f(x) = x \log x$. Write the right-hand-side as

$$\left(\sum_{i=1}^n a_i\right) \frac{\left(\sum_{j=1}^n b_j\right)}{\left(\sum_{i=1}^n a_i\right)} \left(\frac{b_i}{\sum_{j=1}^n b_j} \sum_{i=1}^n \frac{a_i}{b_i}\right) \log \left(\frac{b_i}{\sum_{j=1}^n b_j} \sum_{i=1}^n \frac{a_i}{b_i}\right)$$

- Very handy in proof: e.g., prove $D(p||q) \geq 0$:

$$D(p||q) = \sum p(x) \log \frac{p(x)}{q(x)}$$
$$\geq (\sum_x p(x)) \log \frac{\sum_x p(x)}{\sum_x q(x)} = 1 \log 1 = 0.$$

Convexity of relative entropy

Theorem. $D(p||q)$ *is convex in the pair* (p,q)*: given two pairs of pdf,*

$$D(\lambda p_1 + (1-\lambda)p_2 || \lambda q_1 + (1-\lambda)q_2) \leq \lambda D(p_1||q_1) + (1-\lambda)D(p_2||q_2)$$

for all $0 \leq \lambda \leq 1$.

Proof: By definition and log-sum inequality

$$\begin{aligned}
& D(\lambda p_1 + (1-\lambda)p_2 || \lambda q_1 + (1-\lambda)q_2) \\
&= (\lambda p_1 + (1-\lambda)p_2) \log \frac{\lambda p_1 + (1-\lambda)p_2}{\lambda q_1 + (1-\lambda)q_2} \\
&\leq \lambda p_1 \log \frac{\lambda p_1}{\lambda q_1} + (1-\lambda) \log \frac{(1-\lambda)p_2}{(1-\lambda)q_2} \\
&= \lambda D(p_1||q_1) + (1-\lambda)D(p_2||q_2)
\end{aligned}$$

Concavity of entropy

Entropy

$$H(\boldsymbol{p}) = -\sum_i p_i \log p_i$$

is concave in $\boldsymbol{p}$
Proof 1:

$$\begin{aligned} H(p) &= -\sum_{i\in\mathcal{X}} p_i \log p_i = -\sum_{i\in\mathcal{X}} p_i \log \frac{p_i}{u_i} u_i \\ &= -\sum_{i\in\mathcal{X}} p_i \log \frac{p_i}{u_i} - \sum_{i\in\mathcal{X}} p_i \log u_i \\ &= -D(p||u) - \log \frac{1}{|\mathcal{X}|} \sum_{i\in\mathcal{X}} p_i \\ &= \log|\mathcal{X}| - D(p||u) \end{aligned}$$

Proof 2: We want to prove $H(\lambda p_1 + (1-\lambda)p_2) \geq \lambda H(p_1) + (1-\lambda)H(p_2)$.
A neat idea: introduce auxiliary RV:

$$\theta = \begin{cases} 1, & \text{w. p. } \lambda \\ 2, & \text{w. p. } 1-\lambda. \end{cases}$$

Let $Z = X_\theta$, distribution of Z is $\lambda p_1 + (1-\lambda)p_2$.
Conditioning reduces entropy:

$$H(Z) \geq H(Z|\theta)$$

By their definitions

$$H(\lambda p_1 + (1-\lambda)p_2) \geq \lambda H(p_1) + (1-\lambda)H(p_2).$$

Concavity and convexity of mutual information

Mutual information $I(X;Y)$ is:

(a) concave function of $p(x)$ for fixed $p(y|x)$

(b) convex function of $p(y|x)$ for fixed $p(x)$

Mixing two gases of equal entropy results in a gas with higher entropy.

Proof: write $I(X;Y)$ as a function of $p(x)$ and $p(y|x)$:

$$I(X;Y) = \sum_{x,y} p(x)p(y|x) \log \frac{p(y|x)}{p(y)} =$$

$$= \sum_{x,y} p(x)p(y|x) \log p(y|x) - \sum_{y} \left\{ \sum_{x} p(x)p(y|x) \right\} \log \left\{ \sum_{x} p(y|x)p(x) \right\}$$

(a): Fixing $p(y|x)$, first linear in $p(x)$, second term concave in $p(x)$

(b): Fixing $p(x)$, complicated in $p(y|x)$. Instead of verify it directly, we will relate it to something we know.

Our strategy is to introduce auxiliary RV

$$\tilde{Y}$$

with a mixing distribution

$$p(\tilde{y}|x) = \lambda p_1(y|x) + (1-\lambda)p_2(y|x).$$

To prove convexity, we need to prove:

$$I(X;\tilde{Y}) \leq \lambda I_{p_1}(X;Y) + (1-\lambda)I_{p_2}(X;Y)$$

Since

$$I(X;\tilde{Y}) = D(p(x,\tilde{y})||p(x)p(\tilde{y}))$$

We want to use the fact that $D(p||q)$ is convex in the pair (p,q).

What we need is to find out the pdfs:

$$p(\tilde{y}) = \sum_x [\lambda p_1(y|x)p(x) + (1-\lambda)p_2(y|x)p(x)] = \lambda p_1(y) + (1-\lambda)p_2(y)$$

We also need

$$p(x,\tilde{y}) = p(\tilde{y}|x)p(x) = \lambda p_1(x,y) + (1-\lambda)p_2(x,y)$$

Finally, we get, from convexity of $D(p||q)$:

$$\begin{aligned}
&D(p(x,\tilde{y})||p(x)p(\tilde{y})) \\
=&D(\lambda p_1(y|x)p(x) + (1-\lambda)p_2(y|x)p(x)||\lambda p(x)p_1(y) + (1-\lambda)p(x)p_2(y)) \\
\le&\lambda D(p_1(x,y)||p(x)p_1(y)|) + (1-\lambda)D(p_2(x,y)||p(x)p_2(y)) \\
=&\lambda I_{p_1}(X;Y) + (1-\lambda)I_{p_2}(X;Y)
\end{aligned}$$

Summary of some proof techniques

- Conditioning $p(x, y) = p(x|y)p(y)$, sometimes do this iteratively
- Use Jensen's inequality – identify what is the "average"

$$f(EX) \leq Ef(X)$$

- Prove convexity: several approaches
- Introduce auxiliary random variable – e.g. uniform RV U, indexing RV θ

Summary of important results

- Mutual information is nonnegative
- Conditioning reduces entropy
- Uniform distribution maximizes entropy
- Properties
  - $D(p||q)$ convex in (p, q)
  - Entropy $H(p)$ concave in p
  - Mutual information $I(X;Y)$ concave in $p(x)$ (fixing $p(y|x)$), and convex in $p(y|x)$ (fixing $p(x)$)