Lecture 21: Rate Distortion Theory

- Rate-distortion theorem, and proof ideas
- Calculation of $R(D)$
- Sphere packing for Gaussian source
Rate-distortion theorem

- The **rate distortion region** for a source is the closure of the set of achievable rate distortion pairs \((R, D)\).

- **Rate-distortion function**: \(R(D)\), is the infimum of rates \(R\) such that \((R, D)\) is in the rate distortion region of the source for a given distortion \(D\).

**Theorem.** The rate distortion function for an i.i.d. source \(X\) with distribution \(p(x)\) and bounded \(d(x, \hat{x})\) is equal to

\[
R(D) = \min_{p(\hat{x}|x) : \sum_{(x, \hat{x})} p(x)p(\hat{x}|x)d(x, \hat{x}) \leq D} I(X; \hat{X})
\]
Duality with channel capacity

- \( C = \max_{p(x)} I(X; Y) \)

- \( R = \min_{p(\hat{x} | x) : \sum (x, \hat{x}) p(x)p(\hat{x} | x)d(x, \hat{x}) \leq D} I(X; \hat{X}) \)

- \( I(X; Y) \) is a “function” of \( p(x) \) and \( p(y | x) \)
  - Concave in \( p(x) \) for fixed \( p(y | x) \)
  - Convex in \( p(y | x) \) for fixed \( p(x) \)
Calculation or $R(D)$

Binary Bernoulli($p$) source

$$R(D) = \begin{cases} H(p) - H(D), & 0 \leq D \leq \min\{p, 1-p\} \\ 0, & D > \min\{p, 1-p\} \end{cases}$$

Gaussian source $\mathcal{N}(0, \sigma^2)$

$$R(D) = \begin{cases} \frac{1}{2} \log \frac{\sigma^2}{D}, & 0 \leq D \leq \sigma^2 \\ 0, & D > \sigma^2 \end{cases}$$

Uniform source: $X$ uniformly distributed on $(1, 2, \ldots, m)$

$$R(D) = \begin{cases} \log m - H(D) - D \log(m-1), & 0 \leq D \leq 1 - 1/m \\ 0, & D > 1 - 1/m \end{cases}$$
Spherical packing for Gaussian source

- Gaussian source of variance $\sigma^2$
- $(2^{nR}, n)$ rate distortion code for this source with distortion $D$
- this sequence of code is a set of $2^{nR}$ sequences in $\mathbb{R}^n$ such that most source sequences of length $n$ are within distance $\sqrt{nD}$ of some codeword
- minimum number of codewords required

$$2^{nR(D)} = \left(\frac{\sigma^2}{D}\right)^{n/2}$$

- $R(D) = \frac{1}{2} \log(\sigma^2/D)$
Proof highlights

- Converse: we cannot achieve a distortion of less than $D$ if we describe $X$ at rate less than $R(D)$
  
  Key technique: $R(\lambda D_1 + (1 - \lambda)D_2) \leq \lambda R(D_1) + (1 - \lambda)R(D_2)$

- Achievability: we can find a sequence of code with rate $R(D)$ such that its distortion is less than $D$
  
  Key technique: introduce another typical event:
  
  $$|d(x^n, \hat{x}^n) - Ed(X, \hat{X})| < \epsilon$$

  Random coding, and use joint typicality for decoding