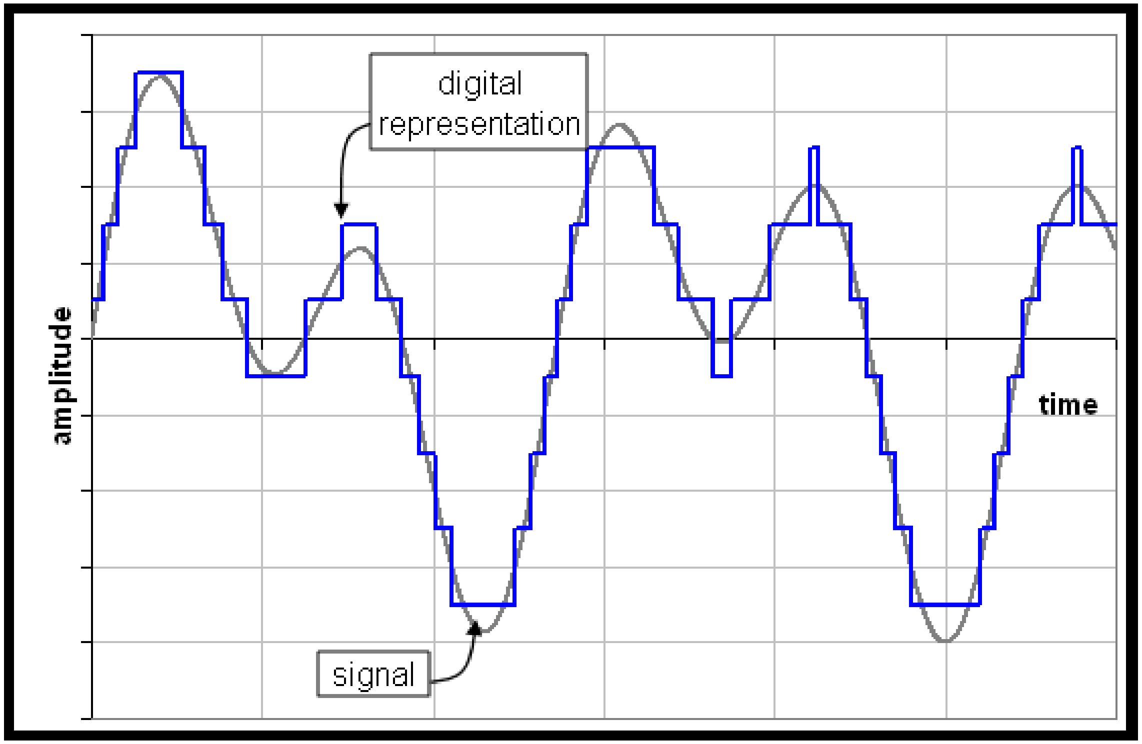
Lecture 20: Quantization and Rate-Distortion

- Quantization
- Introduction to rate-distortion theorem

Approximating continuous signals...



Quantification IV

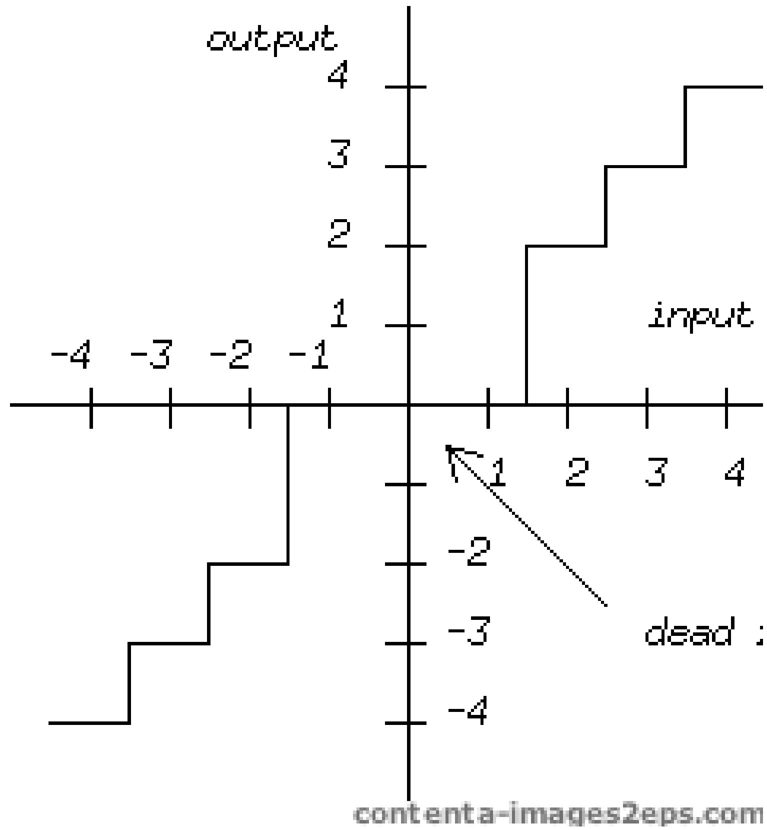
Lossy source coding

- we have seen an information source cannot be **losslessly compressed** beyond its entropy
- in speech, image and video compression, we may tolerate a certain distortion to achieve better compression
- if source is continuous, any compression scheme which translates it into bits will involve distortion
- consider **lossy compression** framework

Quantization

- let X be a continuous random variable
- we approximate X by $\hat{X}(X)$
- using R bits to represent X, then $\hat{X}(X)$ has 2^{nR} possible values
- find the optimal set of values for $\hat{X}$ and associated regions of each value

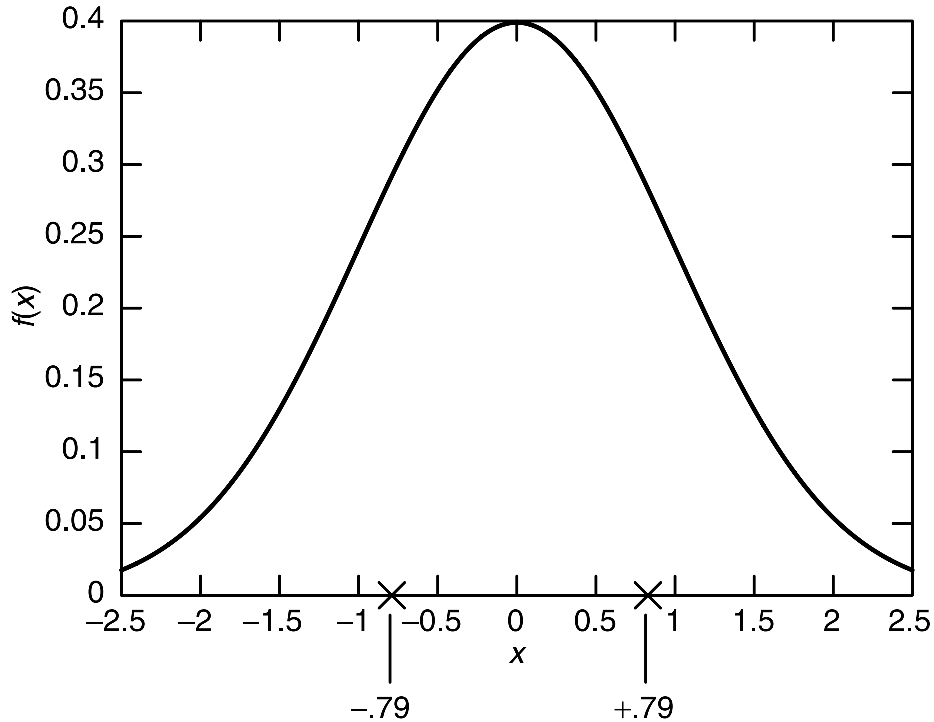
linear scalar quantizer



Example: quantizing Gaussian random variable

- let $X \sim \mathcal{N}(0, \sigma^2)$
- minimize mean square error $E(X - \hat{X}(X))^2$
- if we use 1 bit to represent X, we should let the bit to distinguish the sign of X
- the estimated $\hat{X} = \{E(X|X \geq 0), E(X|X < 0)\}$

$$\hat{X} = \begin{cases} \sqrt{\frac{2}{\pi}}\sigma; & x \geq 0 \\ -\sqrt{\frac{2}{\pi}}\sigma; & x \leq 0 \end{cases}$$



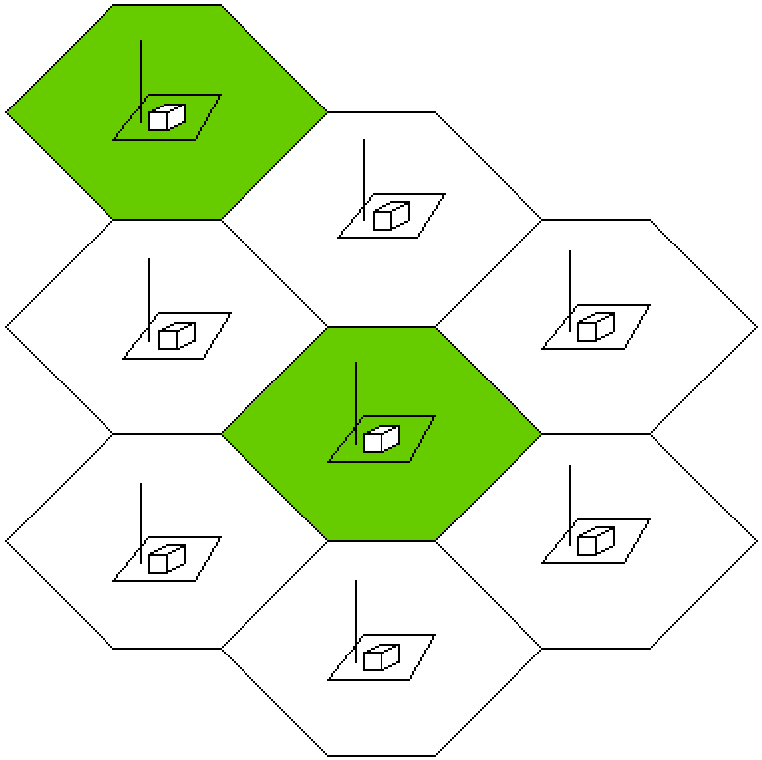
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
f(x)
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5
x
−.79
+.79

- if we are given 2 bits to represent
- we want to divide the real line into 4 regions and use points within each region to represent the sample
- a more complicated optimization problem: boundaries, reconstruction points
- two properties of optimal boundaries and reconstruction points

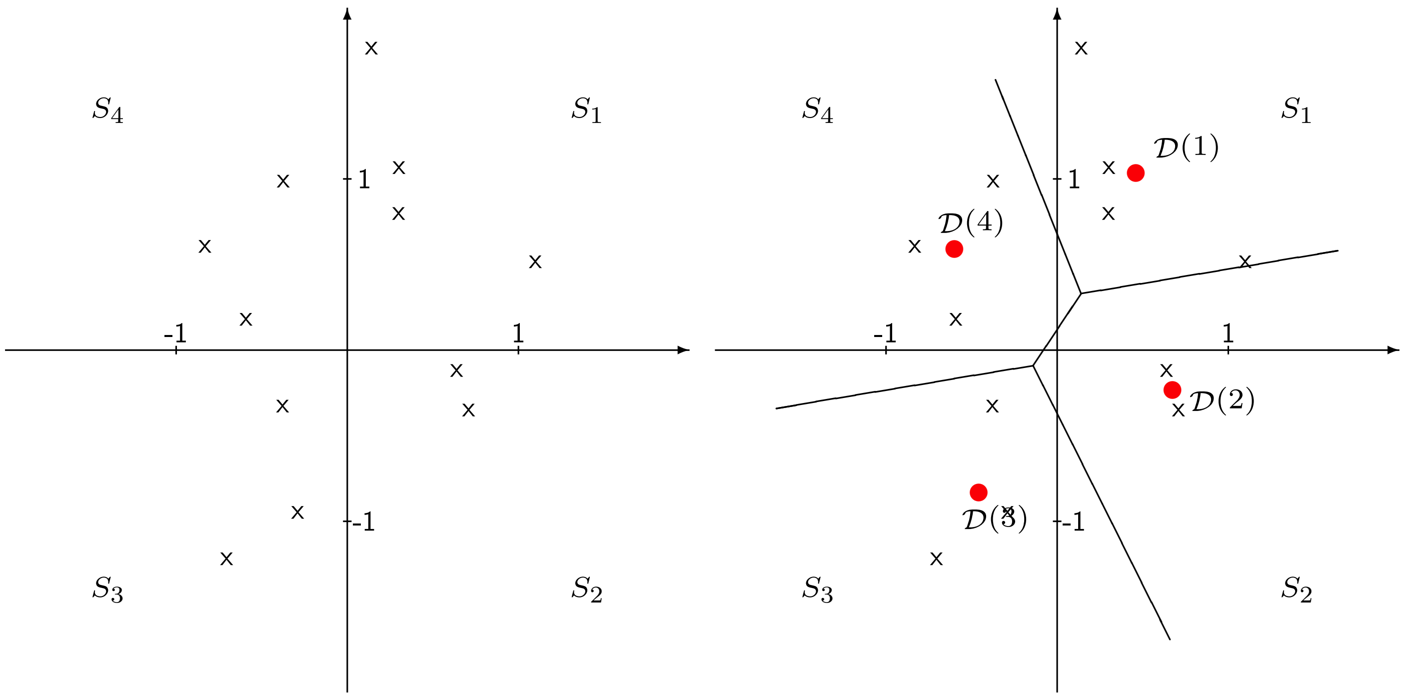
  1) given reconstruction points $\{\hat{X}\}$, distortion is minimized by assigned values to its closest point – *Voronoi or Dirichlet partition*
  2) given partition: reconstruction point should be conditional mean

- iterate these two steps is **Lloyd algorithm**

Voronoi partitions



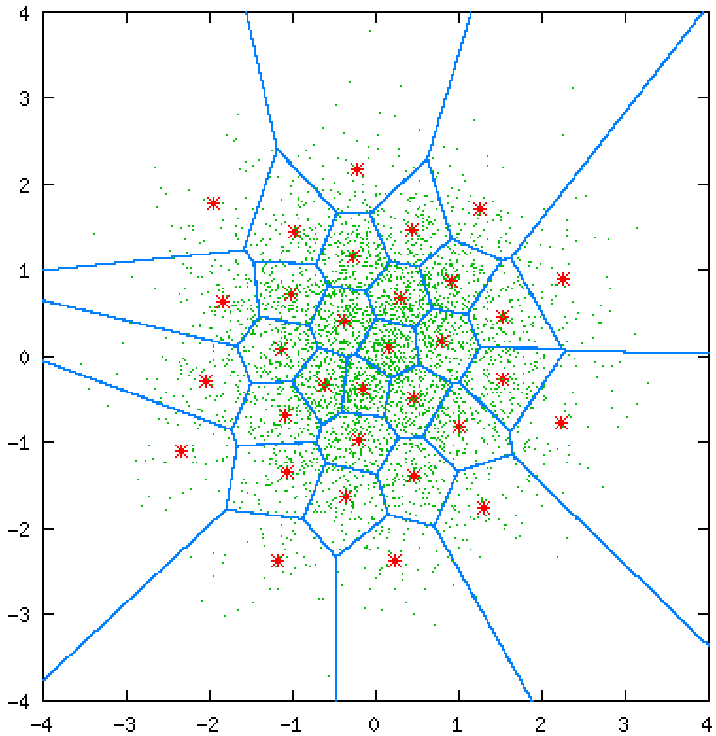
Lloyd algorithm



Vector quantization

- given a set of n samples are i.i.d. from Gaussian
- we want to jointly quantize the vector $[X_1, \ldots, X_n]$
- represent these vectors using nR bits
- represent entire sequence by a single index taking 2^{nR} values
- vector quantization achieve lower distortion than linear quantization

vector quantizer



Rate-distortion tradeoff

- intuition: more bits used, lower quantization error
- can we quantize this tradeoff
- what is the fundamental lower-bound on distortion for a given rate R

Rate-distortion code

- assume a source produces a i.i.d. sequences: $X_1, \ldots, X_n$, $X_n \sim p(x)$
- encoder describes the source sequence X^n by encoding function
- encoding function: $f_n : \hat{X}^n \to \{1, \ldots, 2^{nR}\}$ maps a sequence to an index
- decoder: represent X^n by an estimate $\hat{X}^n$
- decoding function: $g_n : \{1, \ldots, 2^{nR}\} \to \hat{X}^n$ maps an index to reconstructed sequence
- define this $(2^{nR}, n)$-**rate distortion code**

Distortion function

- distortion function: cost of representing symbol by its quantized version

$$d : \mathcal{X} \times \hat{X} \to \mathcal{R}^+$$

- assume $\max_{x\in\mathcal{X},\hat{x}\in\mathcal{X}} d(x,\hat{x}) < \infty$

- example: Hamming distortion $d(x,\hat{x}) = \begin{cases} 0, & x = \hat{x} \\ 1, & x \neq \hat{x} \end{cases}$
  $Ed(X,\hat{X}) = P(X \neq \hat{X})$

- example: squared-error distortion $d(x,\hat{x}) = (x-\hat{x})^2$

- example: Itakura-Saito distance: relative entropy between multivariable normal processes

- for a sequence

$$d(x^n, \hat{x}^n) = \frac{1}{n}\sum_{i=1}^{n} d(x_i, \hat{x}_i)$$

- distortion for a $(2^{nR}, n)$ code:

$$D = Ed(X^n, g_n(f_n(X^n))) = \sum_{x^n} p(x^n) d(x^n, g_n(f_n(x^n)))$$

- a rate distortion pair (R, D) is said to be **achievable** if there exists a sequence of $(2^{nR}, n)$-rate distortion codes (f_n, g_n) with

$$\lim_{n\to} Ed(X^n, g_n(f_n(X^n))) \leq D$$

Rate-distortion theorem

- the **rate distortion region** for a source is the closure of the set of achievable rate distortion pairs (R, D)

- **rate-distortion function**: $R(D)$, is the infimum of rates R such that (R, D) is in the rate distortion region of the source for a given distortion D

Theorem. *The rate distortion function for an i.i.d. source X with distribution $p(x)$ and bounded $d(x, \hat{x})$ is equal to*

$$R(D) = \min_{p(\hat{x}|x): \sum_{(x,\hat{x})} p(x)p(\hat{x}|x)d(x,\hat{x}) \leq D} I(X; \hat{X})$$

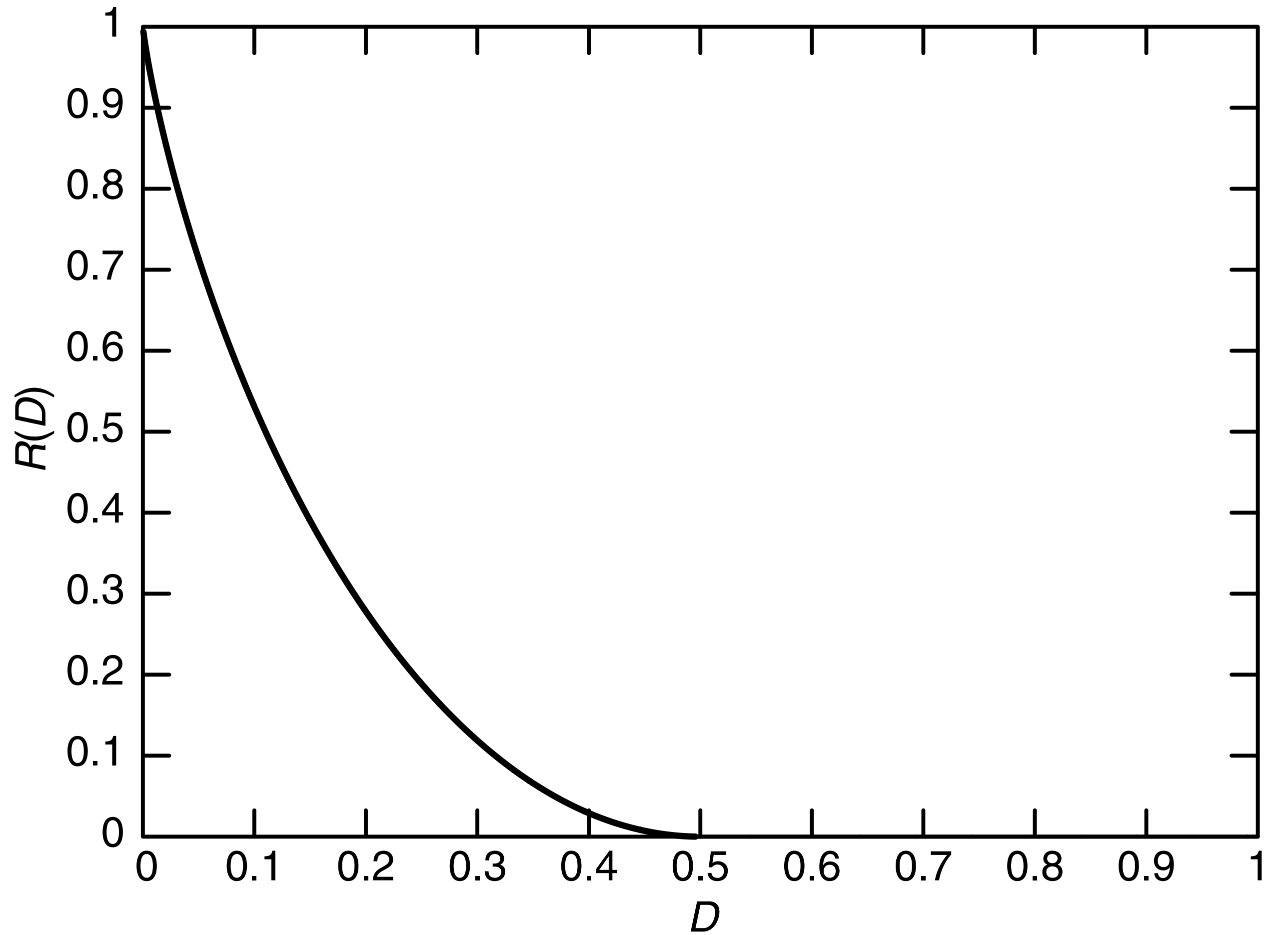


FIGURE 10.4. Rate distortion function for a Bernoulli $(\frac{1}{2})$ source.