Lecture 19: Parallel Gaussian Channels

- Parallel Gaussian channel
- Water-filling
## Christmas gift shopping list

<table>
<thead>
<tr>
<th>Item</th>
<th>Unit Price</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Socks</td>
<td>$2</td>
<td>?</td>
</tr>
<tr>
<td>Tie</td>
<td>$10</td>
<td>?</td>
</tr>
<tr>
<td>Wallet</td>
<td>$5</td>
<td>?</td>
</tr>
<tr>
<td>iPad</td>
<td>$329</td>
<td>?</td>
</tr>
</tbody>
</table>

Budget: $500
Budget allocation problem

• total budget: $w$, money allocated for $n$th item: $w_n$, unit price for $n$th item: $p_n$, can buy $w_n/p_n$ items

• Goal: buy as many gift as possible, but also want to diversify. Diminishing return on the number of items bought $\log(1 + w_n/p_n)$

• budget allocation problem

$$\max_{w_n} \sum_{n=1}^{N} \log(1 + w_n/p_n)$$

subject to $$\sum_{n=1}^{N} w_n = W$$

$$w_n \geq 0$$
Optimal solution: water-filling

- \( w_n = (\nu - p_n)^+ \), \((x)^+ = x \) if \( x \geq 0 \), 0 otherwise

- \( \nu \) determined by budget constraint: \( \sum_{n=1}^{N}(\nu - p_n)^+ = w \)
Parallel Gaussian channels

• Channel capacity of parallel Gaussian channel can be formulated into a similar problem

![Diagram of Parallel Gaussian Channels](image)

**FIGURE 9.3.** Parallel Gaussian channels.
Where are parallel channels?

everywhere:

- OFDM (orthogonal frequency-division multiplexing), parallel channels formed in frequency domain

- MIMO (multiple-input-multiple-output) – multiple antenna system

- DSL (or discrete multi-tone systems)
Parallel independent channels

• $k$ independent channels

• $Y_i = X_i + Z_i$, $i = 1, 2, \ldots, k$, $Z_i \sim \mathcal{N}(0, N_i)$

• total power constraint $E\sum_{i=1}^{k} X_i^2 \leq P$

• goal: distribute power among various channels to maximize the total capacity
Channel capacity

- channel capacity of parallel Gaussian channel

\[
C = \max_{f(x_1, \ldots, x_k): \sum_i EX_i^2 \leq P} I(X_1, \ldots, X_k; Y_1, \ldots, Y_k)
\]

\[
= \frac{1}{2} \log \left(1 + \frac{P_i}{N_i}\right)
\]

- power allocation problem

\[
\max_{P_i} \sum_{i=1}^{k} \log \left(1 + \frac{P_i}{N_i}\right)
\]

subject to \[
\sum_{i=1}^{k} P_i = P
\]

\[
P_i \geq 0
\]
Water-filling solution

\[ \text{Power} \]

\[ \begin{array}{c}
\text{Channel 1} \\
N_1 \\
P_1 \\
\text{Channel 2} \\
P_2 \\
N_2 \\
\text{Channel 3} \\
N_3
\end{array} \]

Dr. Yao Xie, ECE587, Information Theory, Duke University
Channels with colored Gaussian noise

- what if noise in different channels are correlated
- a model for channels with memory
- let $K_z$ be noise covariance matrix
- let $K_x$ be input covariance matrix
- power constraint: $\sum_i E X_i^2 \leq P$, equivalently $\text{tr}(K_x) \leq P$
Channels capacity

- channel capacity is proportional to

\[
\frac{1}{2} \log[(2\pi e)^n |K_x + K_z|]
\]

- input covariance optimization problem

\[
\max_{K_x} \quad \frac{1}{2} \log[(2\pi e)^n |K_x + K_z|]
\]

subject to \( \text{tr}(K_x) = P \)

\( K_x \succeq 0 \)
• solution: $K_x = U \Lambda U^\top$, where $U = \text{eigenvector of } K_z$, and $\Lambda = \text{diagonal matrix}$

• $\lambda_i = (\nu - \lambda_{z,i})^+$, $\lambda_{z,i}$: eigenvalues of $K_z$

• $\nu$ found from: $\sum_{i=1}^{k} (\nu - \lambda_{z,i})^+ = P$
Continuous case

\[ C = \int_{-\pi}^{\pi} \frac{1}{2} \log \left( 1 + \frac{(\nu - N(f))^+}{N(f)} \right) df, \int (\nu - N(f))^+ df = P \]
Summary

Water filling:

- allocate power in parallel Gaussian channels
- optimal power allocation achieve power capacity
- allocate more power in less noisy channels
- very noisy channels are abandoned