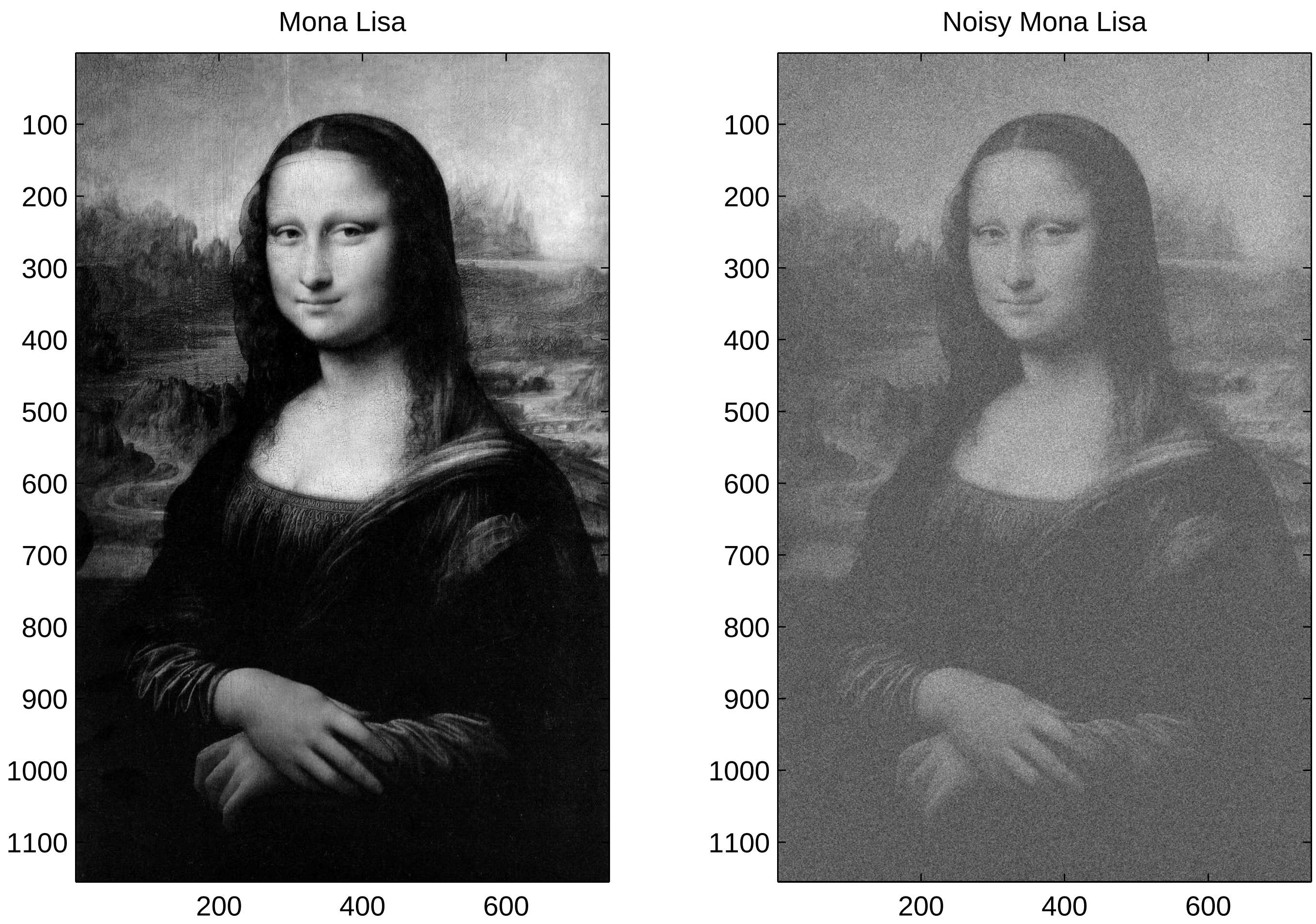
Lecture 18: Gaussian Channel

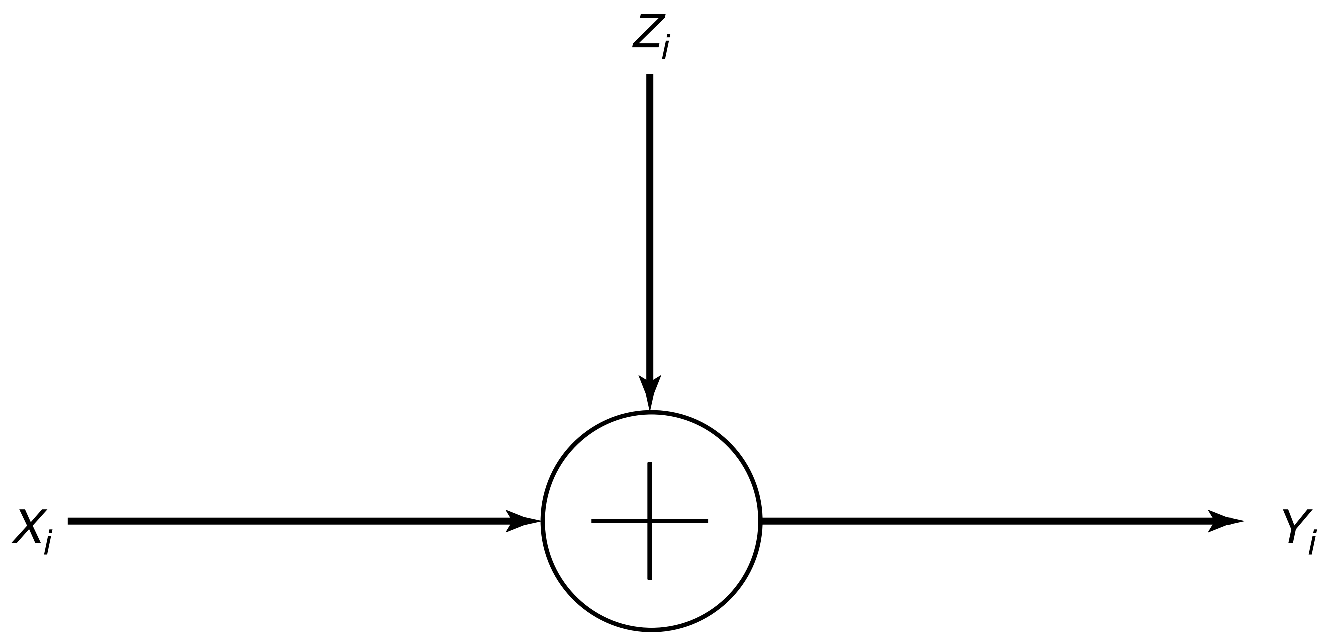
- Gaussian channel
- Gaussian channel capacity

Mona Lisa in AWGN



Gaussian channel

- the most important continuous alphabet channel: AWGN
- $Y_i = X_i + Z_i$, noise $Z_i \sim \mathcal{N}(0, N)$, independent of X_i
- model for communication channels: satellite links, wireless phone



Channel capacity of AWGN

- intuition: $C = \log$ number of distinguishable inputs
- if $N = 0$, $C = \infty$
- if no power constraint on the input, $C = \infty$
- to make it more meaningful, impose *average power constraint*: for any codewords $(x_1, \ldots, x_n)$

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P$$

Naive way of using Gaussian channel

- Binary phase-shift keying (BPSK)
- transmit 1 bit over the channel
- $1 \to +\sqrt{P}$, $0 \to -\sqrt{P}$
- $Y = \pm\sqrt{P} + Z$
- Probability of error

$$P_e = 1 - \Phi(\sqrt{P/N})$$

  normal cumulative probability function (CDF): $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$

- convert Gaussian channel into a discrete BSC with $p = P_e$. Lose information in quantization

Definition: Gaussian channel capacity

- $C = \max_{f(x): EX^2 \le P} I(X;Y)$

- we can calculate from here

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

maximum attained when $X \sim \mathcal{N}(0, P)$

C as maximum data rate

- we can also show this C is the supremum of rate achievable for AWGN
- definition: a rate R is *achievable* for Gaussian channel with power constraint P:

  if there exists a $(2^{nR}, n)$ codes with maximum probability of error $\lambda^n = \max_{i=1}^{2^{nR}} \lambda_i \to 0$ as $n \to \infty$.

Sphere packing

why we may construct $(2^{nC}, n)$ codes with a small probability of error?

Fix one codeword

- consider any codeword of length n
- received vector $\sim \mathcal{N}(\text{true codeword}, N)$
- with high probability, received vector contained in a sphere of radius $\sqrt{n(N+\epsilon)}$ around true codeword
- assign each ball a codeword

Consider all codewords

- with power constraint, with high probability the space of received vector is a sphere with radius $\sqrt{n(P+N)}$

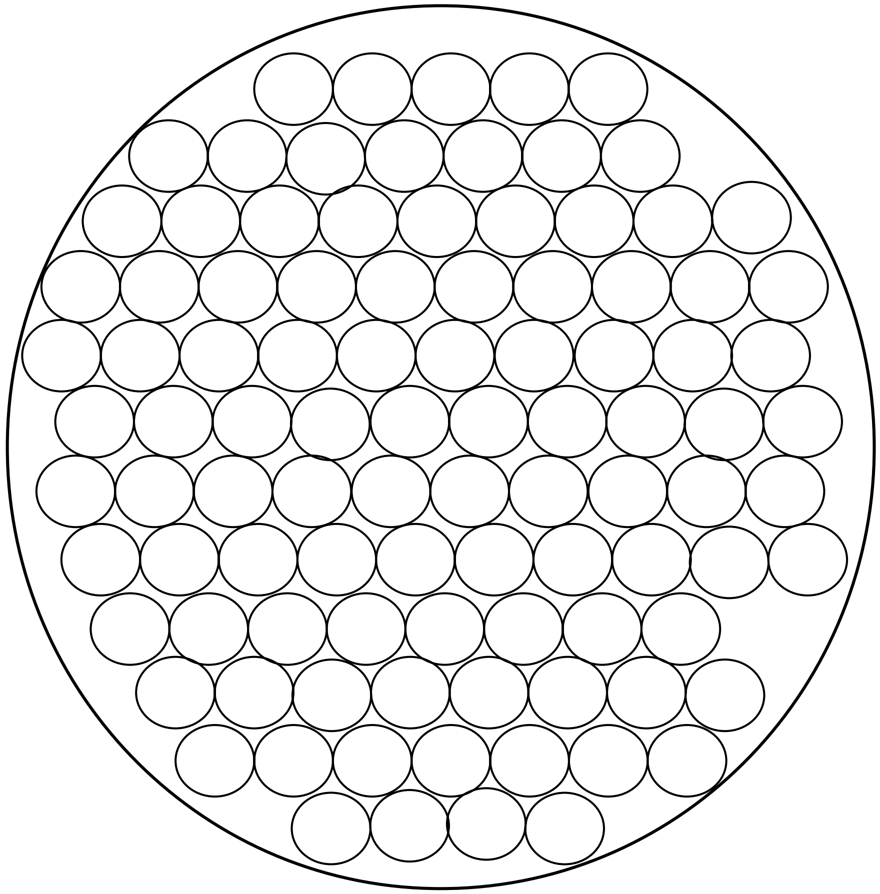
- volume of n-dimensional sphere $= C_n r^n$ for constant C_n and radius r_n

- how many codewords can we pack in a "total power sphere"?

$$\frac{C_n(n(P+N))^{n/2}}{C_n(nN)^{n/2}} = \left(1+\frac{P}{N}\right)^{n/2}$$

- rate of this codebook $= \log_2$(size of the codewords)/n:

$$= \frac{1}{2}\log_2\left(1+\frac{P}{N}\right)$$



sphere packing

Gaussian channel capacity theorem

Theorem. *The capacity of a Gaussian channel with power constraint P and noise variance N is*

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right) \text{ bits per transmission}$$

Proof: 1) achievability; 2) converse

New stuff in proof

Achievability:

- codeword elements generated i.i.d. according $X_j(i) \sim \mathcal{N}(0, P-\epsilon)$. So

$$\frac{1}{n} X_i^2 \to P - \epsilon$$

- power outage error

$$E_0 = \left\{ \frac{1}{n} \sum_{j=1}^{n} X_j^2(1) > P \right\}$$

  small according to law of large number

Converse: Gaussian distribution has maximum entropy

Band-limited channels

- more realistic channel model: band-limited continuous AWGN

$$Y(t) = (X(t) + Z(t)) * h(t)$$

  *: convolution

- Nyquist-Shannon sampling theorem:

  sampling a band-limited signal at sampling rate $1/(2W)$ is sufficient to reconstruct the signal from samples

  "two samples per period"

Capacity of continuous-time band-limited AWGN

- noise has power spectral density $N_0/2$ watts/hertz, bandwidth W hertz, noise power $= N_0 W$
- signal power P watts
- $2W$ samples each second
- channel capacity

$$C = W \log\left(1 + \frac{P}{N_0 W}\right) \text{ bits per second}$$

- when $W \to \infty$, $C \to \frac{P}{N_0} \log_2 e$ bits per second

Telephone line

- telephone signal are band-limited to 3300 Hz
- SNR = 33 dB: $P/(N_0W) = 2000$
- capacity = 36 kb/s
- practical modems achieve transmission rates up to 33.6 kb/s uplink and downlink
- ADSL achieves 56 kb/s downlink (asymmetric data rate)

Summary

- additive white Gaussian noise (AWGN) channel

- noise power: N, signal power constraint P, capacity

$$C = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$$

- band-limited channel with bandwidth $= W$

$$C = W\log\left(1 + \frac{P}{N_0 W}\right)$$