Low-Rank Matrix Recovery With Poisson Noise

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Estimating an image $M^* \in \mathbb{R}_+^{m_1 \times m_2}$ from its linear measurements under Poisson noise is an important problem arises from applications such as optical imaging, nuclear medicine and x-ray imaging [1]. When the image M^* has a low-rank structure, we can use a small number of linear measurements to recover M^*, also known as *low-rank matrix recovery*. This is related to compressed sensing, where the goal is to develop efficient data acquisition systems by exploiting sparsity of underlying signals.

While there has been much success for low-rank matrix recovery and completion under Gaussian noise, little has been done in developing algorithms and establishing performance bounds under Poisson noise. What makes the problems under Poisson noise challenging is that the variance of the noisy observations is proportional to the signal intensity and, hence, instead of using ℓ_2 penalty, we need to use a highly non-linear likelihood function for data fit. Also, in many practical systems associated with Poisson noise, many inherent physical constraints have to be taken into consideration, such as signal positivity and total signal flux preservation.

In this paper, we present a regularized maximum likelihood estimator to recover an approximately low-rank matrix under Poisson noise. We also establish performance bounds for the proposed estimator, by combining techniques for recovering sparse signals under Poisson noise [2], and methods for recovering low-rank matrices [3]. Our bound demonstrates that as the overall intensity of the signal increases, the upper bound on the risk performance of proposed estimator decays at certain rate depending how well the image can be approximated by a low-rank matrix. On the other hand, our bound also indicates there is certain threshold effect such that the risk might not monotonically decrease with respect to the number of measurements, in line with the result in compressed sensing.

Suppose we wish to estimate a signal of image $M^* \in \mathbb{R}_+^{m_1 \times m_2}$ consisting of positive entries. We can make N Poisson measurements $y \in \mathbb{Z}_+^N$, which takes the form of

$$y \sim \text{Poisson}(\mathcal{A}M^*), \tag{1}$$

where the linear operator $\mathcal{A} : \mathbb{R}_+^{m_1 \times m_2} \to \mathbb{R}^N$ models the measurement process of the system. We assume that the total intensity of M^*, given by $I \triangleq \|M\|_{1,1} \triangleq \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} M_{i,j}^*$ is known a priori. Our goal is to estimate the signal M^* from the noisy measurements y.

We propose a regularized maximum-likelihood estimator, which is the solution to the following optimization problem

$$\hat{M} \triangleq \arg\min_{M \in \Gamma} [-\log p(y|\mathcal{A}M) + 2\rho(M)], \tag{2}$$

where $\rho(M) > 0$ is a regularization function. Here Γ is a countable set of feasible estimators with total intensity I

$$\Gamma \triangleq \{M_i \in \mathbb{R}_+^{m_1 \times m_2} : \|M_i\|_{1,1} = I, i = 1, 2, \ldots\}, \tag{3}$$

and the regularization function satisfies the Kraft inequality $\sum_{M \in \Gamma} e^{-\rho(M)} \le 1$. We can think of this formulation as a discretized feasible domain version of the general regularized maximum likelihood estimator. The regularization function assigns small value for low-rank M. Using Kraft-compliant regularization to prefix codes for estimators is a commonly used technique in constructing estimators. Here $\rho(M)$ can be viewed as a measure of complexity for M, or the number of bits needed to represent an estimator M uniquely.

The performance metric we use for estimator is a normalized risk defined as

$$R(M^*, M) \triangleq \frac{1}{I^2} \|M^* - M\|_F^2, \tag{4}$$

where $\|X\|_F$ is the Frobenius norm of a matrix X. We first present a construction for the linear operator $\mathcal{A}$ that satisfies the weak restricted isometry property. For such an $\mathcal{A}$, we establish a performance bound for estimating a general signal M^* from noisy measurements *without* requiring M^* to be low-rank. Then we further make the assumption M^* has a low-rank approximation, and in particular, *compressible*, i.e., its singular value decay geometrically with a rate q. Then we can obtain the following main theorem:

Theorem 1. *Assume $M^* \in \mathbb{R}_+^{m_1 \times m_2}$ is compressible, $[M^*]_{i,j} \ge \frac{c}{N}$ for some positive constant $c \in (0, 1)$. Then exists a finite set of candidate estimators Γ and regularization function satisfying the Kraft inequality such that*

$$\mathbb{E}R(M^*, \hat{M}) \le \mathcal{O}(N) \min_{1 \le k \le k_*} \left[k^{-2\alpha} + \frac{164}{m} + \frac{(m_1 + m_2 + 3)k \log m}{I}\right] + \mathcal{O}\left(\frac{\log(\frac{m_1 m_2}{N})}{N}\right)$$

where the expectation is taken with respect to y, $m \triangleq \min\{m_1, m_2\}$, $k_* \triangleq \frac{2N}{c_1(m_1+m_2+3)\log m}$ for some constant $c_1 > 0$, and $\alpha = 1/q - 1/2$.

References

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