The Diversity-Multiplexing-Delay Tradeoff in MIMO Multihop Networks with ARQ

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Abstract

We study the tradeoff between reliability, data rate, and delay for half-duplex multiple input multiple output (MIMO) multihop networks that utilize the automatic-retransmission-request (ARQ) protocol in the asymptotic high signal-to-noise ratio (SNR) regime. We derive the diversity-multiplexing-delay tradeoff (DMDT) in the high SNR regime, where the delay is caused only by retransmissions. This asymptotic DMDT shows that the performance of an $N$ node network is limited by the weakest three-node sub-network, and the performance of a three-node sub-network is determined by its weakest link, and, hence, the optimal ARQ protocol needs to equalize the performance on each link by allocating ARQ window sizes optimally. This equalization is captured through a novel Variable Block-Length (VBL) ARQ protocol that we propose, which achieves the optimal DMDT.
I. INTRODUCTION

Multihop relays are widely used for coverage extension in wireless networks when the direct link between the source and destination is weak. The coverage of relay networks can be further enhanced by equipping the source, relays and destination with multiple antennas and using multiple-input-multiple-output (MIMO) techniques for beamforming and/or multiplexing. Indeed, MIMO can be used either for beamforming, which improves the reliability, or for spatial multiplexing, which increases the data rate [1]. These dual uses of MIMO gives rise to a diversity-multiplexing tradeoff in point-to-point and multihop MIMO systems, as discussed in more detail below.

Recovery of packets received in error in multihop networks is usually achieved by automatic retransmission (ARQ) protocols. With an ARQ protocol, on each hop, the receiver feeds back to the transmitter a one-bit indicator signifying whether the message can be decoded or not. In case of failure the transmitter retransmits the same message (or incremental information, e.g., using a Raptor code [2][3]) until successful packet reception. The ARQ protocol can be viewed as either a one-bit feedback scheme from the receiver to the transmitter, or as a time diversity scheme employed by the transmitter. The ARQ protocol improves system reliability at a cost of increased delay. In order to design an effective ARQ protocol for multihop relay networks with MIMO nodes, fundamental tradeoff between reliability, data rate, and delay of such systems must be determined, so that the protocol performance can be compared to this theoretical performance limit.

A fundamental tradeoff in designing point-to-point MIMO systems is the tradeoff between reliability and data rate, characterized by the diversity-multiplexing tradeoff (DMT). The asymptotic DMT was introduced in [4] focusing on the asymptotically high SNR regime. The DMT has also been used to characterize the performance of classical three-node relay networks, with a direct link between the source and the destination, when the nodes have single-antenna (SISO) or multiple antennas for various relaying strategies [5], [6], [7]. The DMTs for the amplify-and-forward (AF) and decode-and-forward (DF) relaying strategies are discussed in [5]. Several extensions of the amplify-and-forward strategy have been proposed recently, including the rotate-
and-forward relaying [8] and flip-and-forward relaying [9] strategies, which employ a sequence of forwarding matrices to create an artificial time-varying channel within a single slow fading transmission block in order to achieve a higher diversity gain. A dynamic decode-and-forward (DDF) protocol, in which the relay listens to the source transmission until it can decode the message and then transmits jointly with the source, is proposed in [10] and its DMT performance is shown to dominate the fixed AF and DF schemes. The DDF protocol is shown to achieve the optimal DMT performance in MIMO multihop relay networks in [7]. In this paper, we restrict our attention to multihop networks using the DF relaying strategy, since it enables us to design an optimal ARQ protocol for MIMO multihop relay networks, as we will show later.

Here we consider the diversity-multiplexing-delay tradeoff (DMDT), which was introduced in [11] as an extension of the DMT to include the delay dimension. We consider the delay incurred due to message retransmission over each hop until the message is correctly decoded at the corresponding receiver. Hence the notion of delay in this paper is the maximum number of retransmissions used from the source to the destination. We assume infinite SNR for the asymptotic analysis. With this assumption, [11] presents the DMDT for a point-to-point MIMO system with ARQ, [12] studies the DMDT for cooperative relay networks with ARQ and single-antenna nodes, and [13] proves the DMDT-optimality of ARQ-DDF for the multiple access relay and the cooperative vector multiple access channels with single antenna nodes.

In this paper, we consider a multihop relay network in which a node’s transmission is only received by adjacent nodes in the line. This is a reasonable approximation for environments where received power falls off sharply with distance (i.e., the path loss exponent is large). Our goal is to study the effects of dynamic ARQ on the DMDT in relay networks, which may also be viewed as a discrete protocol implementation of the dynamic decode-and-forward scheme[10][7]. The results presented here are based on our earlier paper [14] for high SNR (the finite SNR regime is explored in [15]). The more general case where non-adjacent nodes receive a given node’s transmission is significantly more complicated, and the optimal DMT is unknown for this case even with a single relay [12].

The contribution of this paper is two-fold: (1) we characterize the DMDT of multihop MIMO
relay networks in the asymptotically high SNR regime; (2) we design the optimal ARQ protocol that achieves the optimal DMDT. Our work extends the DMDT analysis of a point-to-point MIMO system presented in [16] to MIMO multihop relay networks. We derive the DMDT where the delay is caused by retransmissions. For a certain multiplexing gain, the diversity gain is found by studying the information outage probability. An information outage occurs when the receiver fails to decode the message within the maximum number of retransmission rounds allowed. Based on this formulation, for some multihop relay networks a closed-form expression for the DMDT can be identified, whereas for general multihop networks, determining the DMDT can be cast as an optimization problem that can be solved numerically. The DMDT of a general multihop network can be studied by decomposing the network into three-node sub-networks. Each three-node sub-network consists of any three neighboring nodes in the network and the corresponding links between them. The asymptotic DMDT result shows that the performance of the multihop MIMO network, i.e., its DMDT, is determined by the three-node sub-network with the minimum DMDT. The DMDT of the three-node sub-network is again determined by its weakest link. Hence, the optimal ARQ protocol should balance the link DMDT performances on each hop by allocating ARQ window sizes among the hops. From this insight, we present an adaptive variable block-length (VBL) ARQ protocol and prove its DMDT optimality.

The remainder of this paper is organized as follows. Section II introduces the system model and the ARQ protocol. Section III presents the asymptotic DMDT analysis for various ARQ protocols while proving the DMDT optimality of the VBL ARQ. Finally, Section IV concludes the paper and discusses some future directions.

II. SYSTEM MODEL AND ARQ PROTOCOLS

A. Channel Model

Consider an $N$-node multihop MIMO relay network. Node 1 is the source, node $N$ is the destination, while nodes 2 through $N-1$ serve as relays. Node $i$ has $M_i$ antennas for transmitting or receiving, $i = 1, \cdots, N$. The system model is illustrated in Fig. 1. We denote this MIMO relay network as $(M_1, M_2, \cdots, M_N)$. At the source, the message is encoded by a space-time
encoder and mapped into a sequence of $L$ matrices, $\{X_{1,\ell} \in \mathbb{C}^{M_1 \times T} : \ell = 1, \cdots, L\}$, where $T$ is the block length, i.e., the number of channel uses of each block, and $L$ is the maximum number of end-to-end total ARQ rounds that can be used to transmit each message from the source to the destination. The rate of the space-time code is $R$.

We define one ARQ round as the transmitter sending a whole block code of the message to the receiver, which takes $T$ channels uses. We assume that the relays use the DF protocol: node $i$, $2 \leq i \leq N-1$, decodes the message, and reencodes it with a space-time encoder into a sequence of $L$ matrices $\{X_{i,\ell} \in \mathbb{C}^{M_i \times T} : \ell = 1, \cdots, L\}$. The channel between node $i$ and node $(i+1)$ is given by:

$$Y_{i,\ell} = \sqrt{\frac{\text{SNR}}{M_i}} H_{i,\ell} X_{i,\ell} + W_{i,\ell}, \quad 1 \leq \ell \leq L, \quad (1)$$

where $\text{SNR}$ is the average SNR at each antenna, $Y_{i,\ell} \in \mathbb{C}^{M_{i+1} \times T}$, $i = 1, \cdots, N-1$, is the received signal at node $(i+1)$ in the $\ell$th ARQ round. Channels are assumed to be frequency non-selective, block Rayleigh fading and independent of each other, i.e., the entries of the channel matrices $H_{i,\ell} \in \mathbb{C}^{M_{i+1} \times M_i}$ are independent and identically distributed (i.i.d.) complex Gaussian with zero mean and unit variance. The additive noise terms $W_{i,\ell}$ are also i.i.d. circularly symmetric complex Gaussian with zero mean and unit variance. The forward communication links and ARQ feedback links only exist between neighboring nodes.

Other assumptions we have made for the channel model are as follows:

(i) We consider half-duplex relays, that is, the relays cannot transmit and receive at the same time.
(ii) We assume a short-term power constraint at each node for each block code, given by 
\[ \mathbb{E}\{\text{tr}(X_{i,\ell}^\dagger X_{i,\ell})\} \leq M_i, \forall i, \ell. \] Here \( \mathbb{E}\{\cdot\} \) denotes expectation, and \( \dagger \) denotes the Hermitian transpose. A long-term power constraint would allow us to adapt the transmit power and achieve power control gain, as we briefly discuss later in the paper. In the following results we assume a short-term power constraint in order to focus on the diversity gain achieved by the ARQ protocol.

(iii) We consider both the long-term static channel model, in which \( H_{i,\ell} = H_i \) for all \( \ell \), i.e. the channel state remains constant during all the ARQ rounds for a hop and is independent from hop to hop; and the short-term static channel model, where \( H_{i,\ell} \) are i.i.d. but not identical for a given \( \ell \). The long-term static channel assumption is the worst-case in terms of the achievable diversity with a maximum of \( L \) ARQ rounds [11], because there is no time diversity gain. The long-term static channel model may be suitable for modeling indoor applications such as Wi-Fi, while the short-term static channel model suits applications with higher mobility, such as outdoor cellular systems.

**B. Multihop ARQ Protocols**

Consider a family of multihop ARQ protocols, in which the following standard ARQ protocol is used over each hop. The receiver in each hop tries to decode the message after or during one round, depending on whether the synchronization is per-block based or per-channel-use based. Once it is able to decode the message, a one bit acknowledgement (ACK) is fed back to the transmitter that triggers the transmission of the next message. After one ARQ round, if the receiver cannot decode the message, a negative acknowledgement (NACK) is fed back to the transmitter. Then the transmitter sends the next block of the code that carries additional information for the same message. The retransmission over the \( i \)th hop continues for a maximum number of \( L_i \) rounds, called the ARQ window size. Once the ARQ window size is reached without successful decoding of the message, the message is discarded, causing an information outage. Then the next message is transmitted. The sum of the ARQ window sizes is upper
bounded by $L > 0$, where
\[ \sum_{i=1}^{N-1} L_i \leq L. \] (2)

We consider several ARQ protocols with different ways to allocate the available ARQ windows among different hops:

(i) A fixed ARQ protocol, which allocates a fixed ARQ window size of $L_i$ for the transmitter of node $i$, $i = 1, \ldots, N - 1$ such that $\sum_{i=1}^{N-1} L_i = L$.

(ii) An adaptive ARQ protocol, in which the allocation of the ARQ window size per hop is not fixed but adapted to the channel state. The transmitter of a node can keep retransmitting as long as the total ARQ window size of $L$ has not been reached. We further consider two types of adaptive ARQs based on different synchronization levels:

(1) Fixed-Block-Length (FBL) ARQ protocol: The synchronization is per-block based. The transmission of a message over each hop spans an integer number of ARQ rounds.

(2) Variable-Block-Length (VBL) ARQ protocol: The synchronization is per-channel-use based. The receiver can send an ACK as soon as it can decode the message, and the transmitter starts transmitting a new message without waiting until the beginning of the next channel block. VBL has a finer time resolution than FBL and is more efficient in using the available channel block, at a cost of higher synchronization complexity.

We assume that the ARQ feedback links has zero-delay and no error.

III. ASYMPTOTIC DMDT

We characterize the tradeoff among the data rate (measured by the multiplexing gain $r$), the reliability (measured by the diversity gain $d$), and the delay by the asymptotic DMDT of a system with ARQ. Following the framework of [4] and [11], we assume that the rate of transmission depends on the operating SNR, and consider a family of space time codes with block rate $R(SNR)$ scaling with the logarithm of SNR as

\[ R(SNR) = r \log \text{SNR}. \] (3)
In the following, define the exponential equality $\doteq$ as $f(SNR) \doteq SNR^c$, if $\lim_{SNR \to \infty} \frac{\log f(SNR)}{\log SNR} = c$. The exponential inequalities $\leq$ and $\geq$ are defined similarly.

A. Diversity Gain

In the high SNR regime, the diversity gain is defined as the SNR exponent of the message error probability [4]. The diversity gain for a family of codes is defined as:

$$d(r) \triangleq \lim_{SNR \to \infty} \frac{\log P_e(SNR)}{\log SNR}. \quad (4)$$

Define the information outage event as the event that the accumulated mutual information at the receiver within the allowed ARQ window size does not meet the data rate of the message and, therefore, the receiver cannot decode the message. It is shown in [4] that the message error probability $P_e(SNR)$ is dominated by the information outage probability $P_{\text{out}}(SNR)$ in the high SNR regime when the block-length is sufficiently large: $T \geq M_t + M_r - 1$, where $M_t$ and $M_r$ are the number of transmit and receive antennas respectively. In the following we make this assumption. The DMT of an $M_1 \times M_2$ MIMO system is denoted by $d^{(M_1,M_2)}(r)$ and defined as the supremum of the diversity gain $d(r)$ over all families of codes. DMT of a point-to-point MIMO system is characterized in [4] by the following theorem:

**Theorem 1.** Assume the block length $T \geq M_1 + M_2 - 1$. The DMT $d^{(M_1,M_2)}(r)$ is given by the piece-wise linear function connecting the points $(r, (M_1-r)(M_2-r))$, for $r = 0, \ldots, \min(M_1, M_2)$.

Let $C$ be the channel capacity of the $M_1 \times M_2$ MIMO system. From Theorem 1 we know that when the block-length is sufficiently long, for one-to-one MIMO, if one retransmission is allowed

$$P_{\text{out}}(SNR) = P\{C > R\} \doteq SNR^{-d^{(M_1,M_2)}(r)}. \quad (5)$$

B. Asymptotic Equivalences

To characterize the asymptotic DMDT for a multihop network in the high SNR regime, we need the following quantity. Assume that the channel inputs at both the source and the relays are Gaussian with identity covariance matrices. Define $M_i^* = \min\{M_i, M_{i+1}\}$, for $i = 1, \ldots, N-1$. 
For the long-term static channel, let \( \lambda_{i,1}, \ldots, \lambda_{i,M_i^*} \) be the nonzero eigenvalues of \( \mathbf{H}_i \mathbf{H}_i^\dagger \), for \( i = 1, \ldots, N - 1 \). Suppose \( \lambda_{i,j} = \text{SNR}^{-\alpha_{i,j}} \), for \( j = 1, \ldots, M_i^* \), \( i = 1, \ldots, N - 1 \). At high SNR, we can approximate the channel capacities \( C_i(\mathbf{H}_i) = \log \det \left( \mathbf{I} + \frac{\text{SNR}}{M_i} \mathbf{H}_i \mathbf{H}_i^\dagger \right) \) as

\[
C_i(\mathbf{H}_i) \doteq \log \text{SNR}^{S_i(\alpha)} = S_i(\alpha_i) \log \text{SNR},
\]

where

\[
S_i(\alpha_i) \doteq \sum_{j=1}^{M_i^*} (1 - \alpha_{i,j})^+,
\]

\((x)^+ \doteq \max\{x, 0\}\), and the vector \( \alpha_i \doteq [\alpha_{i,1} \cdots \alpha_{i,M_i^*}] \). This \( S_i(\alpha_i) \) plays an important role in the asymptotic DMDT analysis. The closer the SNR exponents \( \alpha_{i,j} \)'s are to unity, the closer the channel matrix is to being singular. Similarly, we can define \( \{\alpha_{i,j}^\ell\} \) in the short-term static channel model and the corresponding matrix \( \alpha_i \in \mathbb{R}^{M_i^* \times L} \) as \( [\alpha_i]_{k,\ell} \doteq \alpha_{i,k}^\ell \).

Proofs for the asymptotic DMDT analysis rely on the notion of decoding time, which is the time at which the accumulated information reaches \( R(\text{SNR}) \). In the case of the short-term static channel, for the FBL ARQ and other block-based ARQ protocols, the decoding time for the \( i \)th node is given by

\[
t_i \doteq \inf \left\{ t \in \mathbb{Z}^+ : \sum_{\ell=1}^{t} C_i(\mathbf{H}_i,\ell) \geq r \log \text{SNR} \right\} = \inf \left\{ t \in \mathbb{Z}^+ : \sum_{\ell=1}^{t} S_i(\alpha_i^\ell) \geq r \right\},
\]

where \( \mathbb{Z}^+ \) denotes the set of positive integers. For the VBL ARQ and other non-block-based ARQ protocols, the decoding time is given by

\[
t_i \doteq \inf \left\{ t \in \mathbb{R} : \sum_{\ell=1}^{[t]} S_i(\alpha_i^\ell) + (t - [t])S_i(\alpha_i^{[t]+1}) \geq r \right\},
\]

where \([x]\) denotes the largest integer smaller than \( x \). Similarly we can define the decoding time for the long-term static channel model. We can view the accumulated mutual information as a random walk with random increments \( S_i(\alpha_i^\ell) > 0 \) and stopping boundary \( r \). Using similar definition, for long-term static channel, the decoding time for FBL ARQ is given by \( t_i = [r/S_i(\alpha_i)] \), and for VRL ARQ \( t_i = r/S_i(\alpha_i) \).
For long-term static channel and VBL ARQ, assuming the number of allowed retransmission for the $i$-th link is $\kappa_i$, define the outage probability on each link as

$$P_{\text{out},i}(\text{SNR}) = P\{\kappa_i C_i(H_i) < R\}. \quad (10)$$

Using (3) and the asymptotic channel capacity approximation (6), under the assumption $L \geq M_i + M_{i+1} - 1$ such that $P_e$ is dominated by the outage probability, the error probability for the $i$-th link is given by

$$P_{e,i}(\text{SNR}) = P_{\text{out},i}(\text{SNR}) = P\{\kappa_i S_i(\alpha_i) \log \text{SNR} < r \log \text{SNR}\} \quad (11)$$

$$= P\{S_i(\alpha_i) < r/\kappa_i\} \quad (12)$$

$$= P\{t_i > \kappa_i\} \quad (13)$$

$$= \text{SNR}^{-d^{(M_i,M_{i+1})}(r/\kappa_i)}. \quad (14)$$

C. Asymptotic DMDT

In the following, we first state our results for the three-node network $(M_1, M_2, M_3)$, and then extend them to the general $N$-node network.

1) Long-Term Static Channel: The DMDT of the fixed ARQ protocol in the case of the long-term static channel is given by the following theorem:

**Theorem 2.** With the long-term static channel assumption, the DMDT of the fixed ARQ protocol for a three-node MIMO multihop network with window sizes $L_1$ and $L_2$, $L_i \in \mathbb{Z}^+$, $L_1 + L_2 \leq L$, is given by:

$$d_{F}^{(M_1,M_2,M_3)}(r, L_1, L_2 | L_1 + L_2 \leq L) = \min_{i=1,2} \left\{ d^{(M_i,M_{i+1})} \left( \frac{r}{L_i} \right) \right\}. \quad (15)$$

Proof: See Appendix A.

Consistent with our intuition, (15) shows that the performance of a three-node network is limited by the weakest link. This implies that if there were no constraints for the $L_i$’s to be integers, the optimal choice should equalize the diversity-multiplexing tradeoff of all the links,
\begin{equation}
    d^{(M_1,M_2)} \left( \frac{r}{L_1} \right) = d^{(M_2,M_3)} \left( \frac{r}{L_2} \right). \tag{16}
\end{equation}

With the integer constraint we choose the integer $L_i$'s such that the minimum of $d^{(M_i,M_{i+1})} \left( \frac{r}{L_i} \right)$ for $i = 1, 2$ is maximized.

The DMDT of the FBL ARQ protocol is a piece-wise linear function characterized by the following theorem:

**Theorem 3.** With the long-term static channel assumption, the DMDT of the FBL ARQ protocol for a three-node MIMO multihop network is given by

\begin{equation}
    d_{FBL}^{(M_1,M_2,M_3)}(r,L) = \min_{\ell_i \in \mathbb{Z}^+: \ell_1 + \ell_2 = L - 1} \left\{ d^{(M_1,M_2)} \left( \frac{r}{\ell_1} \right) + d^{(M_2,M_3)} \left( \frac{r}{\ell_2} \right) \right\}. \tag{17}
\end{equation}

Proof: See Appendix B.

The DMDT of the VBL ARQ protocol cannot always be expressed in closed-form, but can be written as the solution of an optimization problem, as stated in the following theorem.

**Theorem 4.** With the long-term static channel assumption, the DMDT of the VBL ARQ protocol for a three-node MIMO multihop network is given by

\begin{equation}
    d_{VBL}^{(M_1,M_2,M_3)}(r,L) = \inf_{\{\alpha_{i,j}\}} h(\{\alpha_{i,j}\}), \tag{18}
\end{equation}

where

\begin{equation}
    h(\{\alpha_{i,j}\}) \triangleq \frac{2}{\pi} \sum_{i=1}^{M_i^*} \sum_{j=1}^{M_i^*} (2j - 1 + |M_i - M_{i+1}|) \alpha_{i,j}. \tag{19}
\end{equation}

The set $\mathcal{O}$ is defined as

\begin{equation}
    \mathcal{O} \triangleq \{ (\alpha_1, \alpha_2) \in \mathbb{R}^{M_1^*} \times \mathbb{R}^{M_2^*} : \\
    \alpha_{i,1} \geq \cdots \geq \alpha_{i,M_i^*} \geq 0, i = 1, 2, \quad \frac{S_1(\alpha_1)S_2(\alpha_2)}{S_1(\alpha_1) + S_2(\alpha_2)} < \frac{r}{L} \}, \tag{20}
\end{equation}

and this is the optimal DMDT for a three-node network in the long-term static channel.

Proof: See Appendix C.
Note that the DMDT of the VBL ARQ protocol in the three-node network, under the long-term static channel assumption, is similar to the DMT of DDF without ARQ given in [7], with proper scaling of the multiplexing gain. We can show that optimization problem in (18) is equivalent to the following problem (see Appendix D for the derivation):

\[
\begin{align*}
\text{minimize}_{x,y} & \quad d^{(M_1,M_2)}(x/L) + d^{(M_2,M_3)}(y/L) \\
\text{subject to} & \quad \frac{1}{x} + \frac{1}{y} = \frac{1}{r}, \\
 & \quad r \leq x \leq M_1^*, \quad r \leq y \leq M_2^*, \\
\end{align*}
\]

which can be solved numerically. This problem is in general non-convex. However, we can show that at least one of the solutions to \(x\) and \(y\) needs to be an integer point. The intuition is that, if this is not true, then we can always move towards a corner point of the DMT curve and decrease the objective function value. To see this, note that if ignoring the last two inequality constraints, the Lagrangian function is given by

\[
d^{(M_1,M_2)}(x/L) + d^{(M_2,M_3)}(y/L) + \nu(1/x + 1/y - 1/r),
\]

where \(\nu > 0\) is the Lagrangian multiplier. Its derivative with respect to \(x\) is given by

\[
\frac{1}{L} \frac{d^{(M_1,M_2)}}{d^2} (\frac{x}{L}) - \frac{\nu}{x^2} = 0,
\]

and \(\dot{f}\) denotes the derivative of a function \(f\). A similar derivative of the Lagrangian function can be derived \(y\). The derivative \(\dot{d}^{(M_1,M_2)}\) is piecewise constant, since \(d^{(M_1,M_2)}\) is piecewise linear. Hence optimal \(x^*\) and \(y^*\) take the form of

\[
(x^*)^2 = L\nu/c_{x^*}, \quad (y^*)^2 = L\nu/c_{y^*},
\]

where \(c_{x^*} = \dot{d}^{(M_1,M_2)}(x^*/L) > 0\) and \(c_{y^*} = \dot{d}^{(M_2,M_3)}(y^*/L)\) are constants whose value are tuned by \(x^*\) and \(y^*\). Plug this back to the constraint, we have that \(x^*\) and \(y^*\) must satisfy:

\[
\sqrt{c_{x^*}} + \sqrt{c_{y^*}} = \sqrt{L\nu}/r.
\]

Since \(d^{(M_1,M_2)}(x)\) is monotonically decreasing, if \(x\) is not an integer, we can always decrease \(x\) to further reduce the cost function while maintaining the same \(c_x\) such that (23) is still satisfied. Similarly for \(y\). Given this property, we can solve the optimization problem by searching over the corner points of \(x\) or \(y\), depending on decreasing which variable will result in smaller cost value. Based on what we have in 21, if we decrease \(x\), you might end up having \(1/x + 1/y > 1/r\).
Then we will have to increase $y$ then, which you can do until it hits a corner point. This is why “one of them” will be integer, not both (always).

We have closed-form solutions for some specific cases where the optimization problem in (18) has a simple form and can be solved analytically. For example, for a $(M_1, 1, M_3)$ network (18) becomes

$$d_{VBL}^{(M_1, 1, M_3)}(r, L) = \inf_{\alpha_{1,1}, \alpha_{2,1}} \left[ M_1 \alpha_{1,1} + M_3 \alpha_{2,1} \right]$$

subject to

$$\frac{(1 - \alpha_{1,1})^+(1 - \alpha_{2,1})^+}{(1 - \alpha_{1,1})^+(1 - \alpha_{2,1})^+} < \frac{r}{L},$$

$$\alpha_{i,1} \geq 0, \quad i = 1, 2.$$ (24)

The DMDT for this case (and two other special cases) is given by the following corollary:

**Corollary.** With the long-term static channel assumption the DMDT of the VBL ARQ protocol

1) for a $(M_1, 1, M_3)$ MIMO multihop network is given by

$$d_{VBL}^{(M_1, 1, M_3)}(r, L) = \begin{cases} \min\{M_1, M_3\} \frac{1 - 2r/L}{1 - r/L}, & 0 \leq r \leq L/2; \\ 0, & \text{otherwise}. \end{cases}$$ (25)

2) for a $(1, M, 1)$ MIMO multihop network is given by

$$d_{VBL}^{(1, M, 1)}(r, L) = \begin{cases} M_1 \frac{1 - 2r/L}{1 - r/L}, & 0 \leq r \leq L/2; \\ 0, & \text{otherwise}. \end{cases}$$ (26)

3) for a $(2, 2, 2)$ MIMO multihop network is given by

$$d_{VBL}^{(2, 2, 2)}(r, L) = \begin{cases} \frac{2(4 - 5r/L)}{2 - r/L}, & 0 \leq r \leq L/2; \\ \frac{3 - 4r/L}{1 - r/L}, & L/2 \leq r \leq 2L/3; \\ \frac{4(1 - r/L)}{2 - r/L}, & 2L/3 \leq r \leq L; \\ 0, & \text{otherwise}. \end{cases}$$

2) Short-Term Static Channel: The DMDT of the fixed and the FBL ARQ under the short-term static channel assumption are similar to those under the long-term static channel assumption, with additional scaling factors for DMDTs of each hop due to the time diversity gain.
Theorem 5. With the short-term static channel assumption, the DMDT of the FBL ARQ protocol for a three-node MIMO multihop network is given by

\[
d_{FBL}^{(M_1, M_2, M_3)}(r, L) = \min_{\ell_i \in \mathbb{Z}^+ : \ell_1 + \ell_2 = L - 1} \left\{ \sum_{i=1}^{2} \ell_i d(M_i, M_{i+1}) \left( \frac{r}{\ell_i} \right) \right\}.
\]

Proof: See Appendix E.

Theorem 6. With the short-term static channel assumption, the DMDT of the VBL ARQ protocol for a three-node MIMO multihop network is given by

\[
d_{VBL}^{(M_1, M_2, M_3)}(r, L) = \inf_{(\alpha_1, \alpha_2) \in \mathcal{G}} \tilde{h} \left( \{ \alpha_{i,j}^\ell \} \right),
\]

where

\[
\tilde{h} \left( \{ \alpha_{i,j}^\ell \} \right) \triangleq \sum_{i=1}^{2} \sum_{j=1}^{M_i^*} \sum_{\ell=1}^{L} \left( 2j - 1 + |M_i - M_{i+1}| \right) \alpha_{i,j}^\ell,
\]

and the set \( \mathcal{G} \) is defined as

\[
\mathcal{G} \triangleq \left\{ (\alpha_1, \alpha_2) \in \mathbb{R}^{M_1^* \times L} \times \mathbb{R}^{M_2^* \times L} : \alpha_{i,1}^\ell \geq \cdots \geq \alpha_{i,M_i^*}^\ell \geq 0, \forall i, \ell, t_1 + t_2 > L \right\}.
\]

The \( t_i \)'s, defined in (9), depend on the \( \alpha_i \)'s. This is the optimal DMDT for a three-node MIMO multihop network in the short-term static channel.

Proof: See Appendix F.

D. DMDT of an \( N \)-Node Network and Optimality of VBL

Next, we extend our DMDT results to general \( N \)-node MIMO multihop networks. Note that in our model, since each transmitted signal is received only by the next node in the network, the transmission over the \( i \)th hop does not interfere with other transmissions. We will show the DMDT of this more general network is a minimization of the DMDTs of all its three-node sub-networks, due to half-duplexing and multihop diversity. The multihop diversity \cite{7} captures the fact that we allow simultaneous transmissions of multiple node pairs in half-duplex relay networks. For example, while node \( i \) is transmitting to node \( (i + 1) \), node \( (i + 2) \) can also
transmit to node \((i+3)\). This effect allows us to split a message into pieces, which are transmitted simultaneously in the network to increase the multiplexing gain. Using this rate-splitting scheme, we can prove the DMDT optimality of the VBL ARQ protocol. Due to their fixed block length, we are only able to provide upper and lower bounds for DMDTs of fixed ARQ and FBL ARQ in an \(N\)-node network.

**Theorem 7.** With the long-term or short-term static channel assumption, the DMDT of the VBL ARQ for an \(N\)-node MIMO multihop network is given by

\[
d_{VBL}^{(M_1,\cdots,M_N)}(r, L) = \min_{i=1,\cdots,N-2} d_{VBL}^{(M_i,M_{i+1},M_{i+2})}(r, L),
\]

and this is the optimal DMDT for an \(N\)-node network.

**Proof:** See Appendix G.

Theorem 7 says that the DMDT of an \(N\)-node system is determined by the smallest DMDT of its three-node sub-networks. The minimization in Theorem 7 is over all possible three-node sub-networks instead of pairs of nodes, due to the half-duplex constraint: each low-rate piece of message has to wait for the previous piece to go through two hops before it can be transmitted. Theorem 7 also says that the VBL ARQ is the optimal ARQ protocol in the general multihop network.

**Theorem 8.** With the long-term or short-term static channel assumption, the DMDT of fixed ARQ for an \(N\)-node network is lower bounded and upper bounded, respectively, by

\[
d_F^{(M_1,\cdots,M_N)}(r, L_1, \cdots, L_{N-1}) \geq \min_{i=1,\cdots,N-2} d_F^{(M_i,M_{i+1},M_{i+2})} \left( \frac{L_{\max}}{L} r, L_i, L_{i+1} \mid L_i + L_{i+1} \leq L_{\max} \right),
\]

and

\[
d_F^{(M_1,\cdots,M_N)}(r, L_1, \cdots, L_{N-1}) \leq \min_{i=1,\cdots,N-2} d_{VBL}^{(M_i,M_{i+1},M_{i+2})}(r, L),
\]

where \(L_{\max} \triangleq \max_{i=1}^{N-2} \{L_i + L_{i+1}\} \).

**Proof:** See Appendix H.
Theorem 9. With the long-term or short-term static channel assumption, the DMDT of the FBL ARQ for an $N$-node network is lower bounded and upper bounded, respectively, by

\[ d^{[M_1,\cdots,M_N]}_{FBL}(r,L) \geq \min_{i=1,\cdots,N-2} d^{[M_i,M_{i+1},M_{i+2}]}_{VBL}(r,L-N), \]  

and

\[ d^{[M_1,\cdots,M_N]}_{FBL}(r,L) \leq \min_{i=1,\cdots,N-2} d^{[M_i,M_{i+1},M_{i+2}]}_{VBL}(r,L). \]  

\[(33)\]

\[(34)\]

**Proof:** See Appendix I. ■

An intuitive explanation for the DMDT optimality of the VBL ARQ is as follows. Recall that $t_i$ is the number of channel blocks, including retransmissions, needed to decode the message over the $i$th hop. For a three-node network, we can illustrate the information outage region in the region of $t_1 \times t_2$ values as in Fig. 2. The outage region of the VBL ARQ is smaller than those of the fixed and the FBL ARQ. Due to its per-block based synchronization, the outage region boundary of the FBL ARQ is a piecewise approximation to that of the VBL ARQ. In the high SNR regime, we formalize the above intuition in the following corollary to Theorem 9.

![Outage regions](image-url)
Corollary. With the long-term or short-term static channel assumption, for an $N$-node MIMO multihop network, the DMDT of the FBL ARQ converges to that of the VBL ARQ when $L \to \infty$.

Proof: Using (33) and (34), when $L \to \infty$,

$$
\min_{i=1,\ldots,N-2} d^{(M_i,M_{i+1},M_{i+2})}_{VBL}(r,L) \geq \min_{i=1,\ldots,N-2} d^{(M_i,M_{i+1},M_{i+2})}_{FBL}(r,L) \geq \min_{i=1,\ldots,N-2} d^{(M_i,M_{i+1},M_{i+2})}_{VBL}(r,L[1-N/L])
$$

$$
\xrightarrow{L \to \infty} \min_{i=1,\ldots,N-2} d^{(M_i,M_{i+1},M_{i+2})}_{VBL}(r,L).
$$

(35)

E. Power Control Gain with Long Term Power Constraint

With the long-term power constraint and channel state information at the transmitter (CSIT), we can employ a power control strategy to further improve diversity. Let the SNR in the $\ell$th round be $\text{SNR}(\ell) = \text{SNR}_{g(\ell)}$, where $\text{SNR}$ is the average SNR, and $g(\ell)$ is the function defining the power control strategy. In the high SNR regime, similar to (7) we can approximate channel capacities as $C_i(H_i) \approx \log \text{SNR} S'_i(\alpha_i)$, where $S'_i(\alpha_i) = \sum_{j=1}^{M'_i} (g(\ell) - \alpha_{i,j})^+$. Hence, with power control, all the asymptotic DMDT results in the previous sections hold with $S_i(\alpha_i)$ replaced by $S'_i(\alpha_i)$.

F. Examples for Asymptotic DMDT

In this section we show some illustrative examples for the asymptotic DMDT. We first consider the long-term static channel model. For a three-node $(4, 1, 3)$ multihop network with $L = 4$, Fig. 3 shows the DMDT of the fixed ARQ with $L_1 = L_2 = L/2$, of the per-hop-performance-equalizing $L_1$ and $L_2$ satisfying (16), as well as the DMDTs of the FBL and the VBL ARQs. Note that the DMDT of the VBL ARQ in Fig. 3 is the optimal DMDT for the $(4, 1, 3)$ network. We also consider a $(2, 2, 2)$ network, whose DMDTs are shown in Fig. 4.

Fig. 5 presents the three-dimensional DMDT surface of the VBL and the FBL ARQs, respectively, for the $(4, 1, 3)$ multihop network. Note that as $L$ increases, the diversity gain at a given $r$ increases for both the FBL and the VBL ARQ protocols. Also note that the DMDT surface
of the FBL ARQ is piecewise linear and that of the VBL ARQ is smooth due to their different synchronization levels. Fig. 6 illustrates the cross sections of the surfaces in Fig. 5 at $L = 2$ and $L = 10$, which demonstrates the convergence of the DMDTs proved in the corollary of Theorem 9.

Next we consider the short-term static channel model. The DMDT of the (4,1,3) multihop network using the FBL ARQ is shown in Fig. 7. Note that the asymptotic DMDT in the short-term static channel model is not necessarily a multiple $L$ of the corresponding DMDT in the long-term static channel model, which differs from the point-to-point MIMO channel [11], where the asymptotic DMDT in the short-term static channel model is a multiple $L$ of the corresponding
Fig. 5. The three-dimensional DMDT surface for a (4,1,3) network, with the FBL ARQ (left) and the VBL ARQ (right).

Fig. 6. The slices of the DMDT surface in Figure 5 at $L = 2$ (left) and at $L = 10$ (right).

Fig. 7. The DMDT for a (4, 1, 3) multihop network in the long-term static channel versus that in the short-term static channel.
DMDT in the long-term static channel model.

IV. CONCLUSIONS

We have analyzed the asymptotic diversity-multiplexing-delay tradeoff (DMDT) for the $N$-node MIMO multihop relay network with ARQ, under both long-term and the short-term static channel assumptions. We have provided the closed-form asymptotic DMDT expressions are obtained in some special cases. We also proposed the VBL ARQ protocol which adapts the ARQ window size among hops dynamically and proved that it achieves the optimal DMDT under both channel assumptions. We have also shown that the DMDT for general multihop networks with multiple half-duplex relays can be found by decomposing the network into three-node sub-networks such that each sub-network consists of three neighboring nodes and its corresponding two hops. The DMDT of the relay network is determined by the minimum of the DMDTs of the three-node sub-networks. We have also shown that the DMDT of the three-node subnetwork is determined by its weakest link. Hence, the optimal ARQ should equalize the link performances by properly allocating the per-hop ARQ window sizes dynamically.

APPENDIX A

PROOF OF THEOREM 2

With fixed-ARQ protocol and half-duplex relays, the system is in outage if any hop is in outage. The probability of message error $P_e(SNR)$, using the decoding time definition in (8), can be written as:

$$P_e(SNR) \doteq P \{ \{ t_1 > L_1 \} \cup \{ t_2 > L_2 \} \}$$

$$= P \{ t_1 > L_1 \} + P \{ t_2 > L_2 \}$$

$$= P \{ L_1 S_1(\alpha_1) < r \} + P \{ L_2 S_2(\alpha_2) < r \}$$

$$\doteq \sum_{i=1}^{2} SNR^{-d^{(M_i,M_{i+1})}(\frac{r}{\tau_i})}$$

$$\doteq SNR^{-\min_{i=1,2} d^{(M_i,M_{i+1})}(\frac{r}{\tau_i})},$$
where (37) is due to the independence of each link, and (39) follows from (12). The last equality follows since when SNR is high, the dominating term is the one with the larger SNR exponent. Using (40) and the definition of diversity in (4) we obtain the DMDT stated in Theorem 2.

**APPENDIX B**

**PROOF OF THEOREM 3**

For the FBL ARQ protocol with two hops, the probability of message error is given by

$$P_e(\text{SNR}) \doteq P \{t_1 + t_2 > L\} = \sum_{k=1}^{L-1} P \{t_1 = k\} \ P \{t_2 > L - k\} + P \{t_1 \geq L\}.$$  \hfill (41)

In the long-term static channel model we have

$$P \{t_1 = k\} = P \{(k-1)C_1(H_1) < r \log \text{SNR} \leq kC_1(H_1)\}$$

$$= P \left\{C_1(H_1) < \frac{r}{k-1} \log \text{SNR} \right\} - P \left\{C_1(H_1) \leq \frac{r}{k} \log \text{SNR} \right\} \hfill (42)$$

$$= \text{SNR}^{-d(M_1,M_2)(\frac{r}{k-1})} - \text{SNR}^{-d(M_1,M_2)(\frac{r}{k})} = \text{SNR}^{-d(M_1,M_2)(\frac{r}{k-1})},$$

which follows from the fact that \(d(M_1,M_2)(r)\) is monotone decreasing, i.e., \(d(M_1,M_2)(\frac{r}{k-1}) \leq d(M_1,M_2)(\frac{r}{k})\). If we plug (42) into (41) we get

$$P_e(\text{SNR}) \doteq P \left\{C_1(H_1) \geq r \log \text{SNR} \right\} P \left\{C_2(H_2) < \frac{r \log \text{SNR}}{L - 1} \right\}$$

$$+ \sum_{k=2}^{L-1} P \{(L-k)C_2(H_2) < r \log \text{SNR}\} \ P(t_1 = k)$$

$$+ P \left\{C_1(H_1) < \frac{r \log \text{SNR}}{L - 1} \right\} \hfill (43)$$

$$= \text{SNR}^{-\min_{k=2,\ldots,L-1} \left\{d(M_1,M_2)(\frac{r}{k-1})d(M_1,M_2)(\frac{r}{k})+d(M_2,M_3)(\frac{r}{L-k})d(M_2,M_3)(\frac{r}{L-k})\right\}}$$

$$= \text{SNR}^{-\min_{\ell_1+\ell_2=L-1,\ell_i \in \{0,1,\ldots,L-1\}} \left\{d(M_1,M_2)(\frac{r}{\ell_1})+d(M_2,M_3)(\frac{r}{\ell_2})\right\},} \hfill (44)$$

where we have used the fact that \(d(M_i,M_{i+1})(\infty) = 0\). From the definition of diversity in (4) the DMDT in Theorem 3 follows.
APPENDIX C

PROOF OF THEOREM 4

The decoding time of VBL ARQ is real, which differs from FBL ARQ. Using definition for the decoding time, we have

\[ P_e(\text{SNR}) = P\{t_1 + t_2 > L\} = P\left\{ \frac{r}{S_1(\alpha_1) + S_2(\alpha_2)} < \frac{r}{L} \right\}, \]

and

\[ d^{(M_1, M_2, M_3)}_{VBL}(r, L) = \inf_{\{\alpha_{i,j}\} \in \mathcal{O}} h(\{\alpha_{i,j}\}), \]

where \( h(\{\alpha_{i,j}\}) \) is defined in (19), and \( \mathcal{O} \) is defined in (20).

To prove that the DMDT of VBL ARQ is the optimal DMDT in an \( N \)-node network, we first provide an upper bound on the DMDT, and show that the DMDT of the VBL ARQ protocol achieves this upper bound. Our proof is for the short-term static channel in a three-node network, as stated in Theorem 6. A similar (and simpler) proof can be written for the long-term static channel in a three-node network as stated in Theorem 4. Assume that the source transmits for \( \lfloor k \rfloor T \) channel uses (\( \lfloor k \rfloor T < L \)) and the relay transmits in the remaining \( L - \lfloor k \rfloor T \) channel uses. Here \( k \) depends on the channel states and the multiplexing gain \( r \). From the cut-set bound on the multihop network channel capacity, the instantaneous capacity of the MIMO ARQ channel is given by

\[
\min_{k \in (0, L)} \left\{ \max_{P_{X_{1,\ell}} \subseteq H_1, \ell=1,\ldots,\lfloor k \rfloor + 1} \left[ \sum_{\ell=1}^{\lfloor k \rfloor} I(X_{1,\ell}; Y_{1,\ell} | H_{1,\ell}) + (k - \lfloor k \rfloor)I(X_{1,\lfloor k \rfloor + 1}; Y_{1,\lfloor k \rfloor + 1} | H_{1,\lfloor k \rfloor + 1}) \right], \right. \\
\max_{P_{X_{2,\ell}} \subseteq H_2, \ell=\lfloor k \rfloor + 2,\ldots,L} \left[ \sum_{\ell=\lfloor k \rfloor + 2}^{L} I(X_{2,\ell}; Y_{2,\ell} | H_{2,\ell}) + (1 - k + \lfloor k \rfloor)I(X_{2,\lfloor k \rfloor + 1}; Y_{2,\lfloor k \rfloor + 1} | H_{2,\lfloor k \rfloor + 1}) \right] \right\}.
\]

Since the capacity is maximized with Gaussian inputs, and linear scaling of the power constraint
does not affect the high SNR analysis, the capacity $C$ is upper bounded by

$$C \leq \min_{k \in (0, L)} \left\{ \sum_{\ell=1}^{k} C_1(H_{1,\ell}) + (k - \lfloor k \rfloor)C_1(H_{1,[k]+1}), \right. \\
\left. \sum_{\ell=[k]+2}^{L} C_2(H_{2,\ell}) + (1 - k + \lfloor k \rfloor)C_2(H_{2,[k]+1}) \right\}. \quad (47)$$

Now we have, since the channel capacity is upper bounded the minimum of $\sum_{\ell=1}^{L} C_1(H_{1,\ell})$ and $\sum_{\ell=[t_1]+2}^{L} C_2(H_{2,\ell}) + (1 - t_1 + \lfloor t_1 \rfloor)C_2(H_{2,[t_1]+1})$ (given $t_1 < L$), the probability of message error is lower bounded by

$$P_e(SNR) \geq P \left\{ r \log SNR > \sum_{\ell=1}^{L} C_1(H_{1,\ell}), \right. \\
\left. r \log SNR > \sum_{\ell=[t_1]+2}^{L} C_2(H_{2,\ell}) + (1 - t_1 + \lfloor t_1 \rfloor)C_2(H_{2,[t_1]+1}) \right\} \quad (48)$$

$$= P \left\{ \{ t_1 > L \} \cup \{ t_2 > L - t_1 | t_1 < L \} \cap \bar{G} \right\} = P \left\{ \{ t_1 + t_2 > L \} \cap \bar{G} \right\},$$

where $\bar{G} = \left\{ (\alpha_1, \cdots, \alpha_{N-1}) \in \mathbb{R}^{M_1 \times L} \times \cdots \times \mathbb{R}^{M_{N-1} \times L} : \alpha_{i,1}^\ell \geq \cdots \geq \alpha_{i,M_i}^\ell \geq 0, \forall i, \ell \right\}$. Hence, the diversity gain of any ARQ $d(M_1,M_2,M_3)(r,L)$ of a three-node network is upper bounded by

$$d(M_1,M_2,M_3)(r,L) \leq \inf_{\bar{G}_2} \bar{h}(\alpha_{i,j}^\ell), \quad (49)$$

with $\bar{h}(\alpha_{i,j}^\ell)$ defined in (28), and $\bar{G}_2 \triangleq \{ t_1 + t_2 > L \} \cap \bar{G}$, which is the same as the set $\bar{G}$ in (29), the DMDT expression for VBL ARQ in the short-term static channel. This shows that the DMDT upper bound is achieved by the VBL ARQ in the short-term static channel. This completes our proof.
APPENDIX D

DERIVATION OF (21)

Let $x$ and $y$ be two positive constants such that $1/x + 1/y = 1/r$. Note that we can write (45) as

$$\text{SNR}^{-d_{\text{VBL}}(M_1,M_2,M_3,r,L)} \doteq P_e(\text{SNR}) \doteq P \{t_1 + t_2 > L\}$$

$$= P \left\{ \bigcup_{1/x+1/y=1/r} \{t_1 > Lr/x \} \cap \{t_2 > Lr/y \} \right\}$$

$$\geq \sup_{1/x+1/y=1/r} P \left\{ \{t_1 > Lr/x \} \cap \{t_2 > Lr/y \} \right\}$$

$$= \sup_{1/x+1/y=1/r} P \{t_1 > Lr/x \} P \{t_2 > Lr/y \}$$

$$\doteq \sup_{1/x+1/y=1/r} \text{SNR}^{-[d_{\text{VBL}}(M_1,M_2)(x/L) + d_{\text{VBL}}(M_2,M_3)(y/L)]},$$

where we have used the asymptotic equivalence (13). Hence the optimal value of (21) provides an upper-bound on (18).

APPENDIX E

PROOF OF THEOREM 5

In the short-term static channel, for the FBL ARQ protocol with two hops (42) becomes

$$P \{t_1 = k\} = P \left\{ \sum_{\ell=1}^{k-1} C_1(H_{1,\ell}) < r \log \text{SNR} \right\} - P \left\{ \sum_{\ell=1}^{k} C_1(H_{1,\ell}) < r \log \text{SNR} \right\}$$

$$\doteq \left( P \left\{ S_1(\alpha_1^1) < \frac{r \log \text{SNR}}{k-1} \right\} \right)^{k-1} \left( P \left\{ S_1(\alpha_1^1) < \frac{r \log \text{SNR}}{k} \right\} \right)^k$$

$$\doteq \text{SNR}^{-(k-1)d_{\text{VBL}}(M_1,M_2)(r/k)}.$$
Hence, the probability of message error can be written as

\[ P_e(SNR) = P\{C_1(H_{1,1}) \geq r \log SNR\} P\left\{ \sum_{\ell=2}^{L} C_2(H_{2,\ell}) < r \log SNR \right\} \]

\[ + \sum_{k=2}^{L-k} P\left\{ \sum_{\ell=k+1}^{L} C_2(H_{2,\ell}) < r \log SNR \right\} P(t_1 = k) + P\left\{ \sum_{\ell=1}^{L-1} C_1(H_{1,\ell}) < r \log SNR \right\} \]

\[ = SNR^{-\min_{k=2, \ldots, L-1}\{(L-1)d(M_1,M_2)\left(\frac{r}{e-1}\right), (k-1)d(M_1,M_2)\left(\frac{r}{e-1}\right), (L-k)d(M_2,M_3)\left(\frac{r}{e-1}\right), (L-1)d(M_2,M_3)\left(\frac{r}{e-1}\right)\}}. \]

By the definition of diversity, the DMDT in Theorem 5 follows.

**APPENDIX F**

**PROOF OF THEOREM 6**

For a three-node network, we can break down the information outage event as a disjoint union of events that outage happens at the \(i\)th hop:

\[ P_e(SNR) = SNR^{-\inf_{\alpha_{i,j}} \{ \sum_{k=1}^{L} G_k \} G_k} \]

where \( G_k \triangleq \left\{ \sum_{i=1}^{k} t_i > L \right\} \). Due to nonnegativity of \( t_i \), \( G_1 \subset G_2 \). Hence, the minimization should be over \( G_2 \). Adding the ordering requirement on elements of \( \{\alpha_{i,j}\} \), we have Theorem 6.

**APPENDIX G**

**PROOF OF THEOREM 7**

A. **Upper bound**

We will first prove an upper bound for the DMDT for any ARQ protocol in an \(N\)-hop network by considering a genie-aided scheme. For each \( i = 1, \ldots, N-2 \), consider the two consecutive hops from node \( i \) to node \( (i+1) \) and then from node \( (i+1) \) to node \( (i+2) \). Assume a genie aided scheme where the messages are provided to node \( i \), and the output of node \( (i+2) \) is forwarded to the terminal node \( N \). The maximum number of ARQ rounds that can be spent on this two-hop is \( L \). The DMDT of this genie aided setup for any \( i \), is an upper bound on the
The DMDT of the \((M_1, \cdots, M_N)\) system. The optimal DMDT of the \((M_i, M_{i+1}, M_{i+2})\) system with \(L\) ARQ rounds is characterized in Theorem 4. Hence, we have
\[
d^{(M_1, \cdots, M_N)}(r, L) \leq \min_{i=1, \cdots, N-2} d^{(M_i, M_{i+1}, M_{i+2})}_{VBL}(r, L),
\]
where \(d^{(M_1, \cdots, M_N)}(r, L)\) is the DMDT of any ARQ protocol for an \(N\)-hop network.

**B. The DMDT of VBL ARQ**

To be able to exploit the multihop diversity in the network, we use the following rate and ARQ round allocation scheme. First we split the original message of rate \(r \log SNR\) into \(N/2\) lower rate messages each having a rate of \((r \log SNR)/(N/2)\) when \(N\) is even (we split into \((N-1)/2\) lower rate messages when \(N\) is odd). We pump these pieces of the original message into the network sequentially, and in equilibrium, they are transmitted simultaneously by adjacent pairs of nodes.

Moreover, we require the number of blocks allowed for any two-hop transmission, from node \(i\) to node \((i+1)\) and then to node \((i+2)\), for all \(i = 1, \cdots, N-2\), to be \(\bar{L} = L/(N/2)\) when \(N\) is even (or \(\bar{L} = L/[(N-1)/2]\) when \(N\) is odd). This is equivalent to requiring the total number of blocks that each node \(i\), \(i = 2, \cdots, N\), spends for listening and transmitting each piece of a message to be \(\bar{L}\). Note that with this constraint, the end-to-end total number of ARQ rounds used for transmitting each piece of the original message is upper bounded by \(\bar{L} \times N/2 = L\) when \(N\) is even (or equals \(\bar{L} \times (N-1)/2 = L\) when \(N\) is odd). Hence, this scheme satisfies the constraint on the end-to-end total number of ARQ rounds.

It is easy to see that the number of simultaneous transmission pairs we can have in an \(N\) node network is \(N/2\) when \(N\) is even, and \((N-1)/2\) or \((N+1)/2\) when \(N\) is odd.

At the destination, all pieces of a message are combined to decode the original message. From the above analysis, the last piece of these low rate messages is received after at most \(L\) blocks, and the rate of the combined data is \(r \log SNR/(N/2) \times (N/2) = r \log SNR\) when \(N\) is even (similarly for odd \(N\)), which equals the original data rate. Hence this low rate message simultaneous transmission scheme meets both the data rate and end-to-end ARQ window size.
constraints.

Now we study the outage probability $P_{\text{out}}(r)$ of this scheme. Define an outage event for any three-node sub-network consisting of nodes $i$, $(i+1)$, and $(i+2)$, for $N$ even, as:

$$
P_{\text{out}}^i(r, L) \triangleq P \left\{ \frac{r/(N/2) \log \text{SNR}}{C_i(H_i)} + \frac{r/(N/2) \log \text{SNR}}{C_{i+1}(H_{i+1})} > \frac{L}{(N/2)} \right\}
$$

(53)

and for $N$ odd, similarly, as

$$
P_{\text{out}}^i(r, L) \triangleq P \left\{ \frac{r/[(N-1)/2] \log \text{SNR}}{C_i(H_i)} + \frac{r/[(N-1)/2] \log \text{SNR}}{C_{i+1}(H_{i+1})} > \frac{L}{[(N-1)/2]} \right\}.
$$

(54)

Note that (53) and (54) say that by using this scheme, regardless of whether $N$ is even or odd, the outage probability is as if we transmit the original message with data rate $r \log \text{SNR}$ over two hops with a total ARQ round constraint of $L$. From our earlier analysis for the VBL ARQ of a two-hop network, we have that as $\text{SNR} \to \infty$,

$$
P_{\text{out}}^i(r, L) \approxeq \text{SNR}^{-d_{VBL}^{(M_i, M_{i+1}, M_{i+2})}(r, L)}.
$$

(55)

The system is in outage if there is an outage over any of the consecutive two-hop links from the source to the destination. Using a union bound, we have

$$
P_{\text{out}}(r, L) \leq \sum_{i=1}^{N-2} P_{\text{out}}^i(r, L).
$$

(56)

As $\text{SNR}$ goes to infinity, the right hand sum will be dominated by the slowest decaying term, which is the term with minimum $d_{VBL}^{(M_{i}, M_{i+1}, M_{i+2})}(r, L)$. Hence, the DMDT of this scheme is lower bounded by

$$
d^{(M_1, \cdots, M_N)}(r, L) \geq \min_{i=1, \cdots, N-2} d_{VBL}^{(M_i, M_{i+1}, M_{i+2})}(r, L).
$$

(57)

Together with the upper bound in (52), this shows that the presented scheme with the VBL ARQ achieves the optimal DMDT of an $N$-hop network, and its DMDT is given by Theorem 7.
APPENDIX H

PROOF OF THEOREM 8

The proof for the upper bound is identical to the one in Appendix G-A. For the achievable DMDT of the fixed ARQ, we consider the following scheme with deterministic number of ARQ rounds: a node has to wait for at least $L_i$ rounds over hop $i$ for each piece of message, and we allow simultaneous transmissions to employ multihop diversity. Using this scheme, in steady state, the destination will receive one piece of the message every $L_{\text{max}}$ rounds (rather than $L$, if we do not employ multihop diversity). Now we divide the message into pieces with lower rates $\frac{L_{\text{max}}}{L} r \log \text{SNR}$. Using this scheme, overall we will still achieve a rate of $r \log \text{SNR}$ in the steady state by transmitting these lower rate pieces. The outage probability of this scheme provides an upper bound on that of the fixed ARQ protocol:

$$P_{\text{out}}(r, L_1, \cdots, L_{N-1}) \leq \sum_{i=1}^{N-1} P \left\{ \left\lfloor \frac{L_{\text{max}}}{L} r \log \text{SNR} \right\rfloor_{C_i(H_i)} > L_i \right\}, \quad (58)$$

where $\left\lfloor x \right\rfloor$ is the smallest integer larger than $x$. As SNR goes to infinity, the right hand sum will be dominated by the slowest decaying term, which is the term with minimum $d_{F}^{(M_1, \ldots, M_N)} \left( \frac{L_{\text{max}}}{L} r, \frac{r}{L_i} \right)$, and, hence,

$$d_{F}^{(M_1, \ldots, M_N)} (r, L_1, \cdots, L_{N-1}) \geq \min_{i=1, \cdots, N-1} d_{F}^{(M_i, M_i+1)} \left( \frac{L_{\text{max}}}{L} r, \frac{r}{L_i} \right)$$

$$= \min_{i=1, \cdots, N-2} d_{F}^{(M_i, M_i+1, M_{i+2})} \left( \frac{L_{\text{max}}}{L} r, L_i, L_{i+1} \mid L_i + L_{i+1} \leq L_{\text{max}} \right), \quad (59)$$

which completes our proof.

APPENDIX I

PROOF OF THEOREM 9

The proof for the upper bound is identical to Appendix G-A. For the achievable DMDT of the fixed ARQ, again consider the same rate-splitting scheme in Appendix G-B. The difference here is that the number of ARQ rounds used is rounded up to be integer. For $N$ even, the outage
probability can be written as

\[
P^{i}_{\text{out}}(r, \bar{L}) = P \left\{ \left[ \frac{r/(N/2) \log(\text{SNR})}{C_i(H_i)} \right] + \left[ \frac{r/(N/2) \log(\text{SNR})}{C_{i+1}(H_{i+1})} \right] > \bar{L} \right\}
\]

\[
< P \left\{ \frac{r/(N/2) \log(\text{SNR})}{C_i(H_i)} + \frac{r/(N/2) \log(\text{SNR})}{C_{i+1}(H_{i+1})} + 1 > \bar{L} \right\}
\]

\[
= P \left\{ \frac{r \log(\text{SNR})}{C_i(H_i)} + \frac{r \log(\text{SNR})}{C_{i+1}(H_{i+1})} + N > L \right\}
\]

\[
\equiv \text{SNR}^{-d_{VBL}^{M_i,M_{i+1},M_{i+2}}(r,L-N)}.
\]

Note that \( L > N \) since we need at least \( N \) hops. The system is in outage if any three-node sub-network in outage. Using the union bound, again we have

\[
P_{\text{out}}(r, L) \leq \sum_{i=1}^{N-2} P^{i}_{\text{out}}(r, \bar{L}) \leq \sum_{i=1}^{N-2} \text{SNR}^{-d_{VBL}^{M_i,M_{i+1},M_{i+2}}(r,L-N)},
\]

(61)

and

\[
d_{FBL}^{(M_1,\cdots,M_N)}(r, L) \geq \min_{i=1,\cdots,N-2} d_{VBL}^{(M_i,M_{i+1},M_{i+2})}(r,L-N).
\]

(62)

REFERENCES


