MIMO Transmit Beamforming Under Uniform Elemental Power Constraint

Xiayu Zheng, *Student Member, IEEE*, Yao Xie, *Student Member, IEEE*, Jian Li, *Fellow, IEEE*, and Petre Stoica, *Fellow, IEEE*

***Abstract*—We consider multi-input multi-output (MIMO) transmit beamforming under the uniform elemental power constraint. This is a nonconvex optimization problem, and it is usually difficult to find the optimal transmit beamformer. First, we show that for the multi-input single-output (MISO) case, the optimal solution has a closed-form expression. Then we propose a cyclic algorithm for the MIMO case which uses the closed-form MISO optimal solution iteratively. The cyclic algorithm has a low computational complexity and is locally convergent under mild conditions. Moreover, we consider finite-rate feedback methods needed for transmit beamforming. We propose a simple scalar quantization method, as well as a novel vector quantization method. For the latter method, the codebook is constructed under the uniform elemental power constraint and the method is referred as VQ-UEP. We analyze VQ-UEP performance for the MISO case. Specifically, we obtain an approximate expression for the average degradation of the receive signal-to-noise ratio (SNR) caused by VQ-UEP. Numerical examples are provided to demonstrate the effectiveness of our proposed transmit beamformer designs and the finite-rate feedback techniques.**

***Index Terms*—Finite-rate feedback, multi-input multi-output (MIMO), multi-input single-output (MISO), quantization, transmit beamforming, uniform elemental power constraint.**

I. Introduction

EXPLOITING multi-input multi-output (MIMO) spatial diversity is a spectrally efficient way to combat channel fading in wireless communications. Although the theory and practice of receive diversity are well understood, transmit diversity has been attracting much attention only recently. Generally, the transmit diversity systems belong to two groups. In the first group, the channel state information (CSI) is available at the receiver, but not at the transmitter. Orthogonal space-time block codes (OSTBC) [1], [2] have been introduced to achieve the maximum possible spatial diversity order. In the second group, the CSI is exploited at both the transmitter and the receiver via MIMO transmit beamforming, which has recently attracted the attention of the researchers and practitioners alike, due to its much better performance compared to OSTBC [3], [4]. Compared to OSTBC, MIMO transmit beamforming can achieve the same spatial diversity order, full data rate, as well as additional array gains. However, implementing MIMO transmit beamforming schemes in a practical communication system requires additional considerations.

First, optimal transmit beamformers obtained by the conventional, i.e., the maximum ratio transmission (MRT) approach may require different elemental power allocations on the various transmit antennas, which is undesirable from the antenna amplifier design perspective. Especially in an orthogonal frequency division multiplexing (OFDM) system, this power imbalance can result in high peak-to-average power ratio (PAPR), and hencewise reduce the amplifier efficiency significantly [5]. These practical problems have been considered in [6]–[8] for new transmit beamfomer designs, and have also been addressed for transmitter designs in a downlink multiuser system [9].

Second, we need to consider how to acquire the CSI at the transmitter. Recent focus has been on the finite-rate feedback techniques for the current conventional transmit beamforming [10]–[15]. These techniques attempt to efficiently feed back the transmit beamformer (or the CSI) from the receiver to the transmitter via a finite-rate feedback channel, which is assumed to be delay and error free, but bandwidth-limited. The problem is formulated as a vector quantization (VQ) problem [16], [17] and the goal is to design a common codebook, which is maintained at both the transmitter and the receiver. For frequency-flat independently and identically distributed (i.i.d.) Raleigh fading channels, various codebook design criteria can be used and the theoretical performance (e.g., outage probability [12], operational rate-distortion [14], capacity loss [15]) can be analyzed for the multi-input single-output (MISO) case. The feedback schemes can be readily extended to the frequency-selective fading channel case via OFDM. The relationship among the OFDM subcarriers can also be exploited to reduce the overhead of feedback by vector interpolation [18].

We address the aforementioned problems as follows. First, we consider MIMO transmit beamformer design under the uniform elemental power constraint. This is a nonconvex optimization problem, which is usually difficult to solve, and no globally optimal solution is guaranteed [6]. Generally, we can relax the original problem to a convex optimization problem via semidefinite relaxation (SDR). The relaxed problem can be solved via public domain software [19]. We can then obtain a solution to the original nonconvex optimization problem from the solution to the relaxed one by, for example, a heuristic method [20] (referred to as the heuristic SDR solution). Interestingly, we find

Manuscript received October 18, 2006; revised February 19, 2007. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Eric Serpedin. This work was supported in part by the Office of Naval Research under Grant No. N00014-07-1-0193 and the National Science Foundation by Grants ECS-0621879 and CCF-0634786.

X. Zheng and J. Li are with the Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611 USA (e-mail: li@dsp.ufl.edu).

Y. Xie was with the Department of Electrical and Computer Engineering, University of Florida, Gainesville, FL 32611 USA. She is now with the Department of Electrical Engineering, Stanford University, Stanford, CA 94305 USA.

P. Stoica is with the Department of Information Technology, Uppsala University, SE-75105 Uppsala, Sweden.

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSP.2007.896058

out that in the multi-input single-output (MISO) case, the optimal solution has a closed-form expression and is referred to as the closed-form MISO transmit beamformer. (Similar results have appeared in [6]–[8] for equal gain transmission (EGT).) We then propose a cyclic algorithm for the MIMO case which uses the closed-form MISO optimal solution iteratively and the solution is referred to as the cyclic MIMO transmit beamformer. The cyclic algorithm has a low computational complexity and is shown via numerical examples to converge quickly from a good initial point. The numerical examples also show that the proposed transmit beamforming approach outperforms the conventional one with peak power clipping. Meanwhile, the cyclic solution has a comparable performance to the heuristic SDR based design and outperforms the latter when the rank of the channel matrix increases.

Second, we consider finite-rate feedback schemes for the proposed transmit beamformer designs. A simple scalar quantization (SQ) method is proposed; by taking advantage of the property of the uniform elemental power constraint, the number of parameters to be quantized can be reduced to less than one half of their conventional counterpart. VQ methods are also discussed. Although the existing codebooks [10]–[12], [14], [15] can be used with some modifications by the MISO closed-form solution, the performance may not be optimal since they do not take into account the uniform elemental power constraint in the codebook construction. We propose in this paper a VQ method for transmit beamformer designs whose codebook is constructed under the Uniform Elemental Power constraint (referred to as VQ-UEP). The generalized Lloyd algorithm [16] is adopted to construct the codebook. When the number of feedback bits is small, VQ-UEP performs similarly to the conventional VQ (CVQ) method without uniform elemental power constraint. For the MISO case, we further quantify the performance of VQ-UEP by obtaining an approximate closed-form expression for the average degradation of the receive signal-to-noise ratio (SNR). It is shown that this approximate expression is quite tight and that we can use it as a guideline to determine the number of feedback bits needed in practice, for a desired average degradation of the receive SNR.

The remainder of this paper is organized as follows. Section II describes the conventional MIMO transmit beamforming and its limitations. Section III presents our closed-form MISO and cyclic MIMO transmit beamformer designs under the uniform elemental power constraint. In Section IV, we consider the finite-rate feedback schemes, where a simple SQ method and VQ-UEP are proposed. In Section V, we focus on the MISO case and quantify the average degradation of the receive SNR caused by VQ-UEP by obtaining an approximate closed-form expression. Numerical examples are given in Section VI to demonstrate the effectiveness of our designs. We conclude the paper in Section VII. The following notations are adopted throughout this paper.

Notation: Bold upper and lower case letters denote matrices and vectors, respectively. We use $(\cdot)^T$ to denote the transpose and $(\cdot)^*$ to denote the conjugate transpose. $|\cdot|$ stands for the absolute value of a scalar and $\|\cdot\|$ denotes the two-norm of a vector. $\mathcal{C}$ is the complex set; $\mathcal{C}^{M\times N}$ and $\mathcal{R}^{M\times N}$ are the complex- and real-valued $M\times N$ matrices, respectively. $\mathrm{tr}(\cdot)$ is the trace of a matrix. $E\{\cdot\}$ is the expectation, $E_a\{\cdot\}$ is the ensemble average and $\mathrm{Var}\{\cdot\}$ denotes the variance. $\angle\mathbf{x}$ is the vector formed by the phase angles of $\mathbf{x}$ and $\lfloor\cdot\rfloor$ denotes the floor operation.

II. MIMO Transmit Beamforming

Consider an (N_t, N_r) MIMO communication system with N_t transmit and N_r receive antennas in a quasi-static frequency flat fading channel. At the transmitter, the complex data symbol $s \in \mathcal{C}$ is modulated by the beamformer $\mathbf{w}_t = [w_{t,1} \quad w_{t,2} \quad \dots \quad w_{t,N_t}]^T$, and then transmitted into a MIMO channel. At the receiver, after processing with the combining vector $\mathbf{w}_r = [w_{r,1} \quad w_{r,2} \quad \dots \quad w_{r,N_r}]^T$, the sampled combined baseband signal is given by

$$y = \mathbf{w}_r^*[\mathbf{H}\mathbf{w}_t s + \mathbf{n}] \tag{1}$$

where $\mathbf{H} \in \mathcal{C}^{N_r\times N_t}$ is the channel matrix with its (i,j)th element h_{ij} denoting the fading coefficient between the jth transmit and ith receive antennas, and $\mathbf{n} \in \mathcal{C}^{N_r\times 1}$ is the noise vector with its entries being independent and identically distributed (i.i.d.) complex Gaussian random variables with zero-mean and variance σ_n^2. Note that in the presence of interference, i.e., when $\mathbf{n}$ is colored with a known covariance matrix $\mathbf{Q}$, we can use pre-whitening at the receiver to get

$$y = \mathbf{w}_r^*\left[\mathbf{Q}^{-\frac{1}{2}}\mathbf{H}\mathbf{w}_t s + \mathbf{Q}^{-\frac{1}{2}}\mathbf{n}\right]. \tag{2}$$

Hence (2) is equivalent to (1) except that $\mathbf{H}$ in (1) is now replaced by $\mathbf{Q}^{-(1)/(2)}\mathbf{H}$ and the whitened noise has unit variance. Without loss of generality, we focus on (1) hereafter.

The transmit beamformer $\mathbf{w}_t$ and the receive combining vector $\mathbf{w}_r$ in (1) are usually chosen to maximize the receive SNR. Without loss of generality, we assume that $\|\mathbf{w}_t\|^2 = 1$, $\|\mathbf{w}_r\|^2 = 1$, and $E\{|s|^2\} = 1$. Then the receive SNR is expressed as

$$\rho = \frac{E\{|\mathbf{w}_r^*\mathbf{H}\mathbf{w}_t s|^2\}}{E\{|\mathbf{w}_r^*\mathbf{n}|^2\}} = \frac{|\mathbf{w}_r^*\mathbf{H}\mathbf{w}_t|^2}{\sigma_n^2}. \tag{3}$$

To maximize the receive SNR, the optimal transmit beamformer is chosen as the eigenvector corresponding to the largest eigenvalue of $\mathbf{H}^*\mathbf{H}$ [14] (referred to as MRT in [6]), which is also the right singular vector of $\mathbf{H}$ corresponding to its dominant singular value. The optimal combining vector is given by $\mathbf{w}_r = (\mathbf{H}\mathbf{w}_t)/(\|\mathbf{H}\mathbf{w}_t\|)$, which can be shown to be the left singular vector of $\mathbf{H}$ corresponding to its dominant singular value (referred to as maximum ratio combining (MRC) in [6]). Thus, the maximized receive SNR is $\rho = (\lambda_{\max}(\mathbf{H}^*\mathbf{H}))/(\sigma_n^2)$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a matrix. The covariance matrix of the transmitted signal is

$$\mathbf{R} = E\{\mathbf{w}_t s s^* \mathbf{w}_t^*\} = \mathbf{w}_t\mathbf{w}_t^*. \tag{4}$$

The average transmitted power for each antenna is

$$P_i = R_{ii} = |w_{t,i}|^2, \quad i = 1, 2, \dots, N_t \tag{5}$$

where R_{ii} denotes the ith diagonal element of $\mathbf{R}$. (Note that if the constellation of s is phase shift keying (PSK), P_i represents the instantaneous power.) The average power P_i may

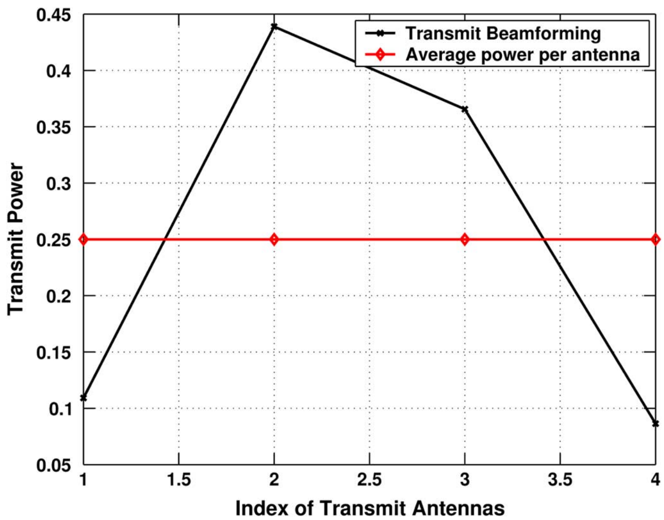


Fig. 1. Transmit power distribution across the index of the transmit antennas for a (4,1) system.

vary widely across the transmit antennas, as illustrated in Fig. 1, which shows a typical example of transmit power distribution across the antennas. The wide power variation poses a severe constraint on power amplifier designs. In practice, each antenna usually uses the same power amplifier, i.e., each antenna has the same power dynamic range and peak power, which means that the conventional MIMO transmit beamforming can suffer from severe performance degradations since it makes the power clipping of the transmitted signals inevitable.

III. Transmit Beamformer Designs Under Uniform Elemental Power Constraint

We consider below both MIMO and its degenerate MISO transmit beamformer designs under the uniform elemental power constraint.

A. Problem Formulation and SDR

Given MRC at the receiver, maximizing the receive SNR ρ in (3) under the uniform elemental power constraint is equivalent to:

$$\begin{array}{ll} & \max_{\{\mathbf{w}_t\}} \ \|\mathbf{H}\mathbf{w}_t\|^2, \\ \text{subject to} & |w_{t,i}|^2 = \frac{1}{N_t}, \quad i = 1, 2, \ldots, N_t. \end{array} \tag{6}$$

This is a nonconvex optimization problem, which is usually difficult to solve, and no globally optimal solution is guaranteed [20], [21], [6]. The problem in (6) can be reformulated as

$$\begin{array}{ll} & \max_{\{\mathbf{R}\}} \ \mathrm{tr}(\mathbf{R}\mathbf{G}), \\ \text{subject to} & R_{ii} = \frac{1}{N_t}, \quad i = 1, 2, \ldots, N_t, \\ & \mathbf{R} \succeq 0, \\ & \mathrm{rank}(\mathbf{R}) = 1, \end{array} \tag{7}$$

where $\mathbf{G} \triangleq \mathbf{H}^*\mathbf{H} \in \mathcal{C}^{N_t \times N_t}$, $\mathbf{R} \in \mathcal{C}^{N_t \times N_t}$, and the inequality $\mathbf{R} \succeq 0$ means that the matrix $\mathbf{R}$ is positive semidefinite. Note that in (7), the objective function is linear in $\mathbf{R}$, the constraints on the diagonal elements of $\mathbf{R}$ are also linear in $\mathbf{R}$, and the positive semidefinite constraint on $\mathbf{R}$ is convex. However, the rank-one constraint on $\mathbf{R}$ is nonconvex. The problem in (7) can be relaxed to a convex optimization problem via SDR, which amounts to omitting the rank-one constraint yielding the following semidefinite program (SDP) [22]:

$$\begin{array}{ll} & \max_{\{\mathbf{R}\}} \ \mathrm{tr}(\mathbf{R}\mathbf{G}), \\ \text{subject to} & R_{ii} = \frac{1}{N_t}, \quad i = 1, 2, \ldots, N_t, \\ & \mathbf{R} \succeq 0. \end{array} \tag{8}$$

The dual form of (8) is given by [20]

$$\begin{array}{ll} & \min_{\{\mathbf{x}\}} \ \mathbf{c}^T\mathbf{x}, \\ \text{subject to} & \mathrm{diag}\{\mathbf{x}\} - \mathbf{G} \succeq 0, \end{array} \tag{9}$$

where $\mathbf{x} \in \mathcal{R}^{N_t \times 1}, \mathbf{c} = (1)/(N_t)\mathbf{1}_{N_t}$ with $\mathbf{1}_{N_t}$ denoting an N_t-dimensional all one column vector, and $\mathrm{diag}\{\mathbf{x}\}$ is a diagonal matrix with $\mathbf{x}$ on its diagonal. The problem in (9) is also a SDP. Both (8) and (9) can be solved by using a public domain SDP solver [19]. The worst case complexity of solving (9) is $\mathcal{O}(N_t^{4.5})$ [23]. We can obtain the optimal solution to (9), whose dual is also the optimal solution to (8).

Assume that the optimal solution to (8) is $\mathbf{R}_{\mathrm{opt}}$. Then $\mathrm{tr}\{\mathbf{R}_{\mathrm{opt}}\mathbf{G}\} \geq \|\mathbf{H}\mathbf{w}_t\|^2$ for any $\mathbf{w}_t$ under the uniform elemental power constraint. If the rank of $\mathbf{R}_{\mathrm{opt}}$ is one, then we obtain the optimal solution $\mathbf{w}_t^{\circ}$ to (6) as the eigenvector corresponding to the nonzero eigenvalue of $\mathbf{R}_{\mathrm{opt}}$. If the rank of $\mathbf{R}_{\mathrm{opt}}$ is greater than one, we can obtain a suboptimal solution $\mathbf{w}_t^{\mathrm{s}}$ from $\mathbf{R}_{\mathrm{opt}}$ via a rank reduction method. For example, the heuristic method in [20] chooses $\mathbf{w}_t^{\mathrm{s}}$ as the eigenvector corresponding to the dorminant eigenvalue of $\mathbf{R}_{\mathrm{opt}}$. The Newton-like algorithm presented in [24] uses the SDR solution as an initial solution and then uses the tangent-and-lift procedure to iteratively find the solution satisfying the rank-one constraint. However, the approximate heuristic method is preferred, as shown in our later discussion, due to its simplicity.

Interestingly, we show below that the optimal solution to (6) has a closed-form expression for the MISO case. Moreover, we propose a cyclic algorithm for the MIMO case which uses the closed-form MISO optimal solution iteratively. The cyclic method has a low complexity and numerical examples in Section VI show that it converges quickly given a good initial point. Furthermore, we also show in Section VI that the performance of the cyclic algorithm is comparable to that of the Heuristic SDR solution and in fact better when the rank of the channel matrix is large. Hence, the former is preferred over the latter in practice.

B. MISO Optimal Transmit Beamformer

Let $\mathbf{h} \in \mathcal{C}^{1 \times N_t}$ be the row channel vector for the MISO case. We consider the maximization problem in (6)

$$\begin{aligned} \|\mathbf{h}\mathbf{w}_t\|^2 &= |\mathbf{h}\mathbf{w}_t|^2 \\ &= \left|\sum_{i=1}^{N_t} h_i w_{t,i}\right|^2 \leq \left[\sum_{i=1}^{N_t} |h_i w_{t,i}|\right]^2 \\ &= \frac{1}{N_t}\left[\sum_{i=1}^{N_t} |h_i|\right]^2, \end{aligned} \tag{10}$$

where the equality holds when $\mathbf{w}_t = (1)/(\sqrt{N_t})e^{j\angle \mathbf{h}^*} \cdot e^{j\phi} \triangleq \mathbf{w}_t^{\circ} \cdot e^{j\phi}$, with $(1)/(\sqrt{N_t})e^{j\angle \mathbf{h}^*}$ denoting the unit-norm column vector having the angles of $\mathbf{h}^*$, and $\phi \in [0, 2\pi)$. Note that the optimal solution is not unique due to the angle ambiguity; yet we may take $\mathbf{w}_t^{\circ}$ as the optimal solution to (6) for simplicity. (This result can also be found in [6]–[8] for EGT.)

C. The Cyclic Algorithm for MIMO Transmit Beamformer Design

The original maximization problem for (6) is

$$\begin{array}{ll} & \max_{\{\mathbf{w}_r,\mathbf{w}_t\}} \ |\mathbf{w}_r^*\mathbf{H}\mathbf{w}_t|^2, \\ \text{subject to} & |w_{t,i}|^2 = \frac{1}{N_t}, \quad i = 1,2,\ldots,N_t, \\ & \|\mathbf{w}_r\| = 1. \end{array} \tag{11}$$

Inspired by the cyclic method (see, e.g., [25]), we solve the problem in (11) in a cyclic way for the MIMO case. The cyclic algorithm is summarized as follows.

1) Step 0: Set $\mathbf{w}_r$ to an initial value (e.g., the left singular vector of $\mathbf{H}$ corresponding to its largest singular value).
2) Step 1: Obtain the beamformer $\mathbf{w}_t$ that maximizes (11) for $\mathbf{w}_r$ fixed at its most recent value. By taking $\mathbf{w}_r^*\mathbf{H}$ as the "effective MISO channel," this problem is equivalent to (6) for the MISO case. The problem is solved in (10) and the optimal solution is:

$$\mathbf{w}_t = (1)/(\sqrt{N_t})e^{j\angle \mathbf{H}^*\mathbf{w}_r}. \tag{12}$$

3) Step 2: Determine the combining vector $\mathbf{w}_r$ that maximizes (11) for $\mathbf{w}_t$ fixed at its most recent value. The optimal $\mathbf{w}_r$ is the MRC and has the form:

$$\mathbf{w}_r = \frac{\mathbf{H}\mathbf{w}_t}{\|\mathbf{H}\mathbf{w}_t\|}. \tag{13}$$

Iterate Steps 1 and 2 until a given stop criterion is satisfied. An important advantage of the above algorithm is that both Steps 1 and 2 have simple closed-form optimal solutions. Also the cyclic algorithm is convergent under mild conditions [25].

We remark here that the cyclic algorithm is flexible and we can add more constraints on $\mathbf{w}_r$ or $\mathbf{w}_t$. A useful one is the uniform elemental power constraint on the receive antennas (or equal gain combining (EGC) [11], [6]), i.e., $|w_{r,i}|^2 = (1)/(N_r), i = 1,2,\ldots,N_r$. Then we only have to modify (13) as $\mathbf{w}_r = (1)/(\sqrt{N_r})e^{j\angle \mathbf{H}\mathbf{w}_t}$ in Step 2 of each iteration. Given a good initial value (e.g., the one as given in Step 0), the cyclic algorithm usually converges in a few iterations in our numerical examples, and the computational complexity of each iteration is very low, involving just (12) and (13).

IV. Finite-Rate Feedback for Transmit Beamforming Designs

In the aforementioned transmit beamformer designs, we have assumed that the transmitter has perfect knowledge on the CSI. However, in many real systems, having the CSI known exactly at the transmitter is hardly possible. The channel information is usually provided by the receiver through a bandwidth-limited finite-rate feedback channel, and SQ or VQ methods, which have been widely studied for source coding [16], [17], can be used to provide the feedback information. To focus on our problem, we assume herein that the receiver has perfect CSI, as usually done in the literatures [10]–[12], [14], [15].

A. Scalar Quantization

Note that the transmit beamformer $\mathbf{w}_t$ under the uniform elemental power constraint can be expressed as

$$\mathbf{w}_t(\theta_0,\ldots,\theta_{N_t-1}) = \frac{1}{\sqrt{N_t}}\begin{bmatrix} e^{j\theta_0} \\ \vdots \\ e^{j\theta_{N_t-1}} \end{bmatrix} \tag{14}$$

where the transmit beamformer $\mathbf{w}_t(\theta_0,\ldots,\theta_{N_t-1})$ is a function of N_t parameters $\{\theta_i, \theta_i \in [0,2\pi)\}_{i=0}^{N_t-1}$. Via simple manipulations, we obtain

$$\begin{aligned} \mathbf{w}_t(\theta_0,\ldots,\theta_{N_t-1}) &= \frac{1}{\sqrt{N_t}}e^{j\theta_0}\begin{bmatrix} 1 \\ e^{j\tilde{\theta}_1} \\ \vdots \\ e^{j\tilde{\theta}_{N_t-1}} \end{bmatrix} \\ &\triangleq e^{j\theta_0}\mathbf{w}_t(\tilde{\theta}_1,\ldots,\tilde{\theta}_{N_t-1}) \end{aligned} \tag{15}$$

where $\tilde{\theta}_i = \theta_i - \theta_0, \tilde{\theta}_i \in [0,2\pi), i = 1,2,\ldots N_t-1$. Since $\|\mathbf{H}\mathbf{w}_t(\theta_0,\ldots,\theta_{N_t-1})\|^2 = \|\mathbf{H}\mathbf{w}_t(\tilde{\theta}_1,\ldots,\tilde{\theta}_{N_t-1})\|^2$, we can reduce one parameter and quantize $\mathbf{w}_t(\tilde{\theta}_1,\ldots,\tilde{\theta}_{N_t-1})$ instead of $\mathbf{w}_t(\theta_0,\ldots,\theta_{N_t-1})$.

Denote

$$\mathbf{w}_t\left(\tilde{\theta}_1^{n_1},\ldots,\tilde{\theta}_{N_t-1}^{n_{N_t-1}}\right) = \frac{1}{\sqrt{N_t}}\begin{bmatrix} 1 \\ e^{j\tilde{\theta}_1^{n_1}} \\ \vdots \\ e^{j\tilde{\theta}_{N_t-1}^{n_{N_t-1}}} \end{bmatrix} \tag{16}$$

where $\tilde{\theta}_i^{n_i} = (2\pi n_i)/(N_i)$, $0 \le n_i \le N_i - 1, i = 1,2,\ldots,N_t-1$, with $N_i = 2^{B_i}$ and n_i denoting the number of quantization levels and feedback index of $\tilde{\theta}_i$, respectively, and where B_i is the number of feedback bits for $\tilde{\theta}_i$. After obtaining the transmit beamformer from (10) or the cyclic algorithm in Section III.C, we quantize the parameters $\tilde{\theta}_i$ to the "closest" (via round off) grid points $\tilde{\theta}_i^{n_i}$, $i = 1,2,\ldots,N_t-1$. Hence for this scalar quantization scheme, we need to send the index set $(n_1,n_2,\ldots,n_{N_t-1})$ from the receiver to the transmitter, which requires $B = \sum_{i=1}^{N_t-1} B_i$ bits. The receive combing vector is $\mathbf{w}_r = (\mathbf{H}\mathbf{w}_t(\tilde{\theta}_1^{n_1},\ldots,\tilde{\theta}_{N_t-1}^{n_{N_t-1}}))/(\|\mathbf{H}\mathbf{w}_t(\tilde{\theta}_1^{n_1},\ldots,\tilde{\theta}_{N_t-1}^{n_{N_t-1}})\|)$.

The choice of $\{B_i\}_{i=1}^{N_t-1}$ is known as a counting problem [26], which has $C_{B+N_t-2}^{B} = ((B+N_t-2)!)/(B!(N_t-2)!)$ combinations. The optimal set $\{B_i\}_{i=1}^{N_t-1}$ is the one that maximizes $\|\mathbf{H}\mathbf{w}_t(\tilde{\theta}_1^{n_1},\ldots,\tilde{\theta}_{N_t-1}^{n_{N_t-1}})\|^2$. However, this exhaustive search is too complicated for practical applications. One simple suboptimal approach is to make B_i approximately equal. Specifically, let $B_{a_1} = \lfloor (B)/(N_t - 1)\rfloor, B_{a_2} = B_{a_1} + 1$ and $N_s = B - B_{a_1}(N_t - 1)$. Then we can let $B_i = B_{a_2}$ bits for the first N_s parameters $\{\tilde{\theta}_i\}_{i=1}^{N_s}$ and $B_i = B_{a_1}$ bits for the remaining $(N_t-1) - N_s$ parameters $\{\tilde{\theta}_i\}_{i=N_s+1}^{N_t-1}$.

We remark here that for the conventional MIMO transmit beamformer without uniform elemental power constraint, the SQ requires about twice as many parameters. In this case, the transmit beamformer is expressed as

$$\mathbf{w}_t(A_0,\theta_0,\ldots,A_{N_t-1},\theta_{N_t-1}) = \begin{bmatrix} A_0e^{j\theta_0} \\ \vdots \\ A_{N_t-1}e^{j\theta_{N_t-1}} \end{bmatrix} \tag{17}$$

where $A_i, A_i \in [0,1]$ is the ith amplitude and $\theta_i, \theta_i \in [0, 2\pi)$ is the ith phase of the transmit beamformer vector, respectively, and hence there are totally $2N_t$ parameters.

B. Ad-Hoc Vector Quantization

Vector quantization can be adopted to further reduce the feedback overhead. In this case, both the transmitter and the receiver have to maintain a common codebook with a finite number of codewords. The codebook can be constructed based on several criteria. One approach is to directly apply the existing codebooks (e.g., [10]–[12], [14], [15]) constructed for the conventional transmit beamformer designs obtained without the uniform elemental power constraint. Among them, the criteria (e.g., [10], [14], [15]) that can be implemented by the generalized Lloyd algorithm can always lead to a monotonically convergent codebook. The generalized Lloyd algorithm is based on two conditions: the nearest neighborhood condition (NNC) and the centroid condition (CC) [16], [14], [15]. NNC is to find the optimal partition region for a fixed codeword, while CC updates the optimal codeword for a fixed partition region. The monotonically convergent property is guaranteed due to obtaining an optimal solution for each condition. Maximizing the average receive SNR is a widely used criterion to design the codebook [10], [12], [14] and will also be adopted here for codebook construction. Some modifications are still needed as below when the uniform elemental power constraint is imposed.

Let a codebook constructed for the conventional transmit beamforming be $\tilde{\mathcal{W}} := \{\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2, \ldots, \tilde{\mathbf{w}}_{N_v}\}$, where $N_v = 2^B$ is the number of codewords in the codebook $\tilde{\mathcal{W}}$, and B is the number of feedback bits. The receiver first chooses the optimal codeword in the codebook as:

$$\tilde{\mathbf{w}}^{\mathrm{o}} = \arg\max_{\tilde{\mathbf{w}} \in \tilde{\mathcal{W}}} \|\mathbf{H}\tilde{\mathbf{w}}\|^2 \tag{18}$$

where the operator $\arg\max$ returns a global maximizer. Then we need to feed back the index of $\tilde{\mathbf{w}}^{\mathrm{o}}$ from the receiver to the transmitter, which requires B bits. The transmit beamformer satisfying the uniform elemental power constraint is obtained as:

$$\hat{\mathbf{w}}_{\mathrm{ad}} = \frac{1}{\sqrt{N_t}} e^{j\angle \tilde{\mathbf{w}}^{\mathrm{o}}} \tag{19}$$

and the receive combining vector is $\mathbf{w}_r = (\mathbf{H}\hat{\mathbf{w}}_{\mathrm{ad}})/(\|\mathbf{H}\hat{\mathbf{w}}_{\mathrm{ad}}\|)$. However, the codebook $\tilde{\mathcal{W}}$ may not be optimal for our proposed transmit beamformer designs, since it is ad-hocly constructed without the uniform elemental power constraint (referred to as the ad-hoc vector quantization (AVQ) method).

C. Vector Quantization Under Uniform Elemental Power Constraint

Like AVQ, herein we also maximize the average receive SNR, while the codebook is constructed under the uniform elemental power constraint (referred to as "VQ-UEP"). For a given codebook $\hat{\mathcal{W}} := \{\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \ldots, \hat{\mathbf{w}}_{N_v}\}$, the receiver first chooses the optimal transmit beamformer as:

$$\hat{\mathbf{w}}_{\mathrm{opt}} = \arg\max_{\hat{\mathbf{w}} \in \hat{\mathcal{W}}} \|\mathbf{H}\hat{\mathbf{w}}\|^2 \tag{20}$$

and the corresponding vector quantizer is denoted as $\hat{\mathbf{w}}_{\mathrm{opt}} = \mathcal{Q}(\mathbf{H})$. Then we need to feedback the index of $\hat{\mathbf{w}}_{\mathrm{opt}}$ from the receiver to the transmitter with $\log_2 N_v = B$ bits, and the receive combining vector is $\mathbf{w}_r = (\mathbf{H}\hat{\mathbf{w}}_{\mathrm{opt}})/(\|\mathbf{H}\hat{\mathbf{w}}_{\mathrm{opt}}\|)$.

Now the design problem becomes finding the codebook, which can be constructed off-line as follows. First, we generate a training set $\{\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_{\hat{N}}\}$ from a sufficiently large number $\hat{N}$ of channel realizations. Next, starting from an initial codebook (e.g., a codebook obtained from the conventional transmit beamformer designs or one obtained via the splitting method [16]), we iteratively update the codebook according to the following two criteria until no further improvement is observed.

1) NNC: for given codewords $\{\hat{\mathbf{w}}_i\}_{i=1}^{N_v}$, assign a training element $\mathbf{H}_n$ to the ith region

$$S_i = \{\mathbf{H}_n : \|\mathbf{H}_n\hat{\mathbf{w}}_i\|^2 \geq \|\mathbf{H}_n\hat{\mathbf{w}}_j\|^2, \forall j \neq i\} \tag{21}$$

where $S_i, i = 1, 2, \ldots, N_v$, is the partition set for the ith codeword $\hat{\mathbf{w}}_i$.

2) CC: for a given partition S_i, the updated optimum codewords $\{\hat{\mathbf{w}}_i\}_{i=1}^{N_v}$ satisfy

$$\hat{\mathbf{w}}_i = \arg\max_{\|\hat{\mathbf{w}}_i\|=1} E_a\left[\|\mathbf{H}_n\hat{\mathbf{w}}_i\|^2 | \mathbf{H}_n \in S_i\right]$$
$$\text{subject to} \quad |\hat{w}_{i,m}|^2 = \frac{1}{N_t}, \quad m = 1, \ldots, N_t \tag{22}$$

for $i = 1, 2, \ldots, N_v$. Let $\mathbf{R}_i = E_a[\mathbf{H}_n^*\mathbf{H}_n | \mathbf{H}_n \in S_i]$ and $\mathbf{R}_i^{1/2}$ be Hermitian square root of $\mathbf{R}_i$. A simple reformulation results in

$$\hat{\mathbf{w}}_i = \arg\max_{\|\hat{\mathbf{w}}_i\|=1} \left\|\mathbf{R}_i^{1/2}\hat{\mathbf{w}}_i\right\|^2$$
$$\text{subject to} \quad |\hat{w}_{i,m}|^2 = \frac{1}{N_t}, \quad m = 1, \ldots, N_t. \tag{23}$$

This problem is identical to (6) ($\mathbf{H}$ is replaced by $\mathbf{R}_i^{1/2}$) and can be efficiently solved by the cyclic algorithm proposed in Section III.C.

V. Average Degradation of the Receive SNR

For frequency flat i.i.d. MISO Rayleigh fading channels, various analysis approaches have been proposed to quantify the vector quantization effect (outage probability [12], operational rate-distortion [14], capacity loss [15], etc.). These analyses provide theoretical insights into the vector quantization methods and can serve as a guideline for determining the optimum number of feedback bits needed for the conventional transmit beamforming. We quantify below the effect of VQ-UEP with finite-bit feedback on our closed-form MISO transmit beamformer design. Let $\mathbf{h} \sim \mathcal{N}(0, \sigma_h^2 \mathbf{I}_{N_t})$. Without loss of generality, we assume $(E\{|s|^2\})/(\sigma_n^2) = 1$. The average degradation of the receive SNR is defined as:

$$\begin{aligned} D_v &= E\left\{|\mathbf{h}\mathbf{w}_t^\circ|^2 - |\mathbf{h}\mathcal{Q}(\mathbf{h})|^2\right\} \\ &= E\left\{|\mathbf{h}\mathbf{w}_t^\circ|^2\right\} \\ &\quad - \sum_{i=1}^{N_v} P(\mathbf{h} \in \bar{S}_i) E\left\{|\mathbf{h}\hat{\mathbf{w}}_i|^2 | \mathbf{h} \in \bar{S}_i\right\}, \\ &= E\left\{|\mathbf{h}\mathbf{w}_t^\circ|^2\right\} \\ &\quad - \sum_{i=1}^{N_v} P(\mathbf{v}_t \in \tilde{S}_i) E\left\{|\mathbf{v}_t^* \hat{\mathbf{w}}_i|^2 | \mathbf{v}_t \in \tilde{S}_i\right\} E\{\|\mathbf{h}\|^2\}, \end{aligned} \tag{24}$$

where $\bar{S}_i = \{\mathbf{h} : |\mathbf{h}\hat{\mathbf{w}}_i|^2 \geq |\mathbf{h}\hat{\mathbf{w}}_j|^2, \forall j \neq i\}$ is the partition set (or Voronoi cell) for the ith codeword $\hat{\mathbf{w}}_i, \tilde{S}_i = \{\mathbf{v}_t : \mathbf{v}_t = (\mathbf{h}^*)/(\|\mathbf{h}\|), \mathbf{h} \in \bar{S}_i\}, P(\mathbf{h} \in \bar{S}_i)$ is the probability that a channel realization $\mathbf{h}$ belongs to the partition $\bar{S}_i$, and the last equality is due to the independence between $\|\mathbf{h}\|$ and the normalized vector $\mathbf{h}/\|\mathbf{h}\|$ [14], [26]. Obviously, we have $P(\mathbf{h} \in \bar{S}_i) = P(\mathbf{v}_t \in \tilde{S}_i)$.

A. *Maximum Average Receive SNR* $E\{|\mathbf{h}\mathbf{w}_t^\circ|^2\}$

Using $\mathbf{w}_t^\circ$ in (10), we get:

$$\begin{aligned} E\left\{|\mathbf{h}\mathbf{w}_t^\circ|^2\right\} &= \frac{1}{N_t} E\left\{(|h_1| + |h_2| + \cdots + |h_{N_t}|)^2\right\} \\ &= \frac{1}{N_t}\left[\mathrm{Var}\left\{\sum_{i=1}^{N_t} |h_i|\right\} + E^2\left\{\sum_{i=1}^{N_t} |h_i|\right\}\right] \\ &= \mathrm{Var}\{|h_1|\} + N_t E^2\{|h_1|\}, \end{aligned} \tag{25}$$

where the last equality is due to the i.i.d. property of $\{|h_i|\}_{i=1}^{N_t}$. The $|h_1|$ in (25) has the probability density function (pdf) as follows [27]:

$$f_{|h_1|}(x) = \frac{2x}{\sigma_h^2} \exp\left\{-\frac{x^2}{\sigma_h^2}\right\}, \quad x > 0. \tag{26}$$

The mean and variance of $|h_1|$ are, respectively

$$E\{|h_1|\} = \sqrt{\frac{\pi\sigma_h^2}{4}} \tag{27}$$

and

$$\mathrm{Var}\{|h_1|\} = \left(1 - \frac{\pi}{4}\right)\sigma_h^2. \tag{28}$$

Combining (27) and (28) into (25), we get

$$\begin{aligned} E\left\{|\mathbf{h}\mathbf{w}_t^\circ|^2\right\} &= \left(1 - \frac{\pi}{4}\right)\sigma_h^2 + N_t\left(\sqrt{\frac{\pi\sigma_h^2}{4}}\right)^2 \\ &= \sigma_h^2 + \frac{\pi}{4}(N_t - 1)\sigma_h^2. \end{aligned} \tag{29}$$

B. *Approximate Value of* $E\{|\mathbf{v}_t^*\hat{\mathbf{w}}_i|^2 | \mathbf{v}_t \in \tilde{S}_i\}$

Note that the vector $\mathbf{v}_t$ is considered as uniformly distributed on the unit hypersphere Ω^{N_t} [10], [12], [14], [15]. For a fixed codeword $\hat{\mathbf{w}}_i \in \Omega^{N_t}$, the random variable $\gamma_i = |\mathbf{v}_t^*\hat{\mathbf{w}}_i|^2$ has a beta distribution $\mathrm{Beta}(1, N_t - 1)$ [15], with the pdf:

$$f_{\gamma_i}(x) = (N_t - 1)(1 - x)^{N_t - 2}, \quad 0 < x < 1. \tag{30}$$

Now we consider the conditional density $f_{\gamma_i|\mathbf{v}_t \in \tilde{S}_i}(x)$. Generally, each Voronoi cell [10], [12], [15], [16] obtained from the generalized Lloyd algorithm has a very complicated shape and it is difficult to obtain an exact closed-form expression for $f_{\gamma_i|\mathbf{v}_t \in \tilde{S}_i}(x)$. We adopt herein the approximate method used in [12], [15] to analyze the problem at our hand.

When N_v is reasonably large, we can approximate the probability $P(\mathbf{v}_t \in \tilde{S}_i)$ as $P(\mathbf{v}_t \in \tilde{S}_i) \simeq (1)/(N_v), \forall i$. The Voronoi cells can be considered as identical to each other. We then approximate each Voronoi cell $\tilde{S}_i$ as a spherical segment on the surface of a unit hypersphere:

$$\tilde{S}_i \simeq \hat{S}_i = \left\{\mathbf{v}_t \in \Omega^{N_t} : \delta \leq |\mathbf{v}_t^*\hat{\mathbf{w}}_i|^2 \leq a\right\} \tag{31}$$

where $a \triangleq (E_{\mathbf{h}}\{|\mathbf{h}\mathbf{w}_t^\circ|^2\})/(E_{\mathbf{h}}\{\|\mathbf{h}\|^2\}) = (1)/(N_t) + (\pi)/(4)(N_t - 1)/(N_t)$ is the maximum average value of $|\mathbf{v}_t^*\hat{\mathbf{w}}_i|^2$ achieved by perfect feedback in our MISO transmit beamformer design, and the parameter $\delta > 0$ is the minimum value of $|\mathbf{v}_t^*\hat{\mathbf{w}}_i|^2$ in each Voronoi cell. We need to solve the following equation related to B to obtain δ:

$$\begin{aligned} P(\mathbf{v}_t \in \tilde{S}_i) &= P(\delta \leq \gamma_i \leq a) \\ &= \int_\delta^a f_{\gamma_i}(x)dx = \frac{1}{2^B}. \end{aligned} \tag{32}$$

Using the pdf in (30), we get

$$\delta = 1 - [2^{-B} + (1 - a)^{N_t - 1}]^{\frac{1}{N_t - 1}}. \tag{33}$$

Thus, for the Voronoi cell $\tilde{S}_i$, we approximate the conditional pdf of γ_i as

$$\begin{aligned} f_{\gamma_i|\mathbf{v}_t \in \tilde{S}_i}(x) &= \frac{f_{\gamma_i}(x)}{P(\mathbf{v}_t \in \tilde{S}_i)}, \quad \delta \leq x < a, \\ &= 2^B(N_t - 1)(1 - x)^{N_t - 2} 1_{[\delta, a)}(x) \end{aligned} \tag{34}$$

where

$$1_{[\delta, a)}(x) = \begin{cases} 1, & \delta \leq x < a, \\ 0, & \text{otherwise} \end{cases} \tag{35}$$

is the indication function.

From the conditional pdf $f_{\gamma_i|\mathbf{v}_t \in \tilde{S}_i}(x)$ in (34), we obtain

$$\begin{aligned} &E\left\{|\mathbf{v}_t^*\hat{\mathbf{w}}_i|^2 | \mathbf{v}_t \in \tilde{S}_i\right\} \\ &= \int_0^1 x f_{\gamma_i|\mathbf{v}_t \in \tilde{S}_i}(x)\, dx \\ &= \int_\delta^a x \cdot 2^B (N_t - 1)(1 - x)^{N_t - 2} dx \\ &= 1 + \frac{N_t - 1}{N_t} \\ &\quad \cdot 2^B[(1 - a)^{N_t} - (1 - \delta)^{N_t}]. \end{aligned} \tag{36}$$

C. Quantifying the Average Degradation of the Receive SNR

Now we quantify the average degradation of the receive SNR in (24) using the approximate conditional pdf $f_{\gamma_i|\mathbf{v}_t \in \tilde{S}_i}(x)$. From (36), we observe that the average receive SNR γ_0 is

$$\begin{aligned}\gamma_0(B) &= \sum_{i=1}^{N_v} P(\mathbf{v}_t \in \tilde{S}_i) E \\ &\quad \times \left\{ |\mathbf{v}_t^* \hat{\mathbf{w}}_i|^2 | \mathbf{v}_t \in \tilde{S}_i \right\} E\{\|\mathbf{h}\|^2\} \\ &= \sum_{i=1}^{N_v} \frac{1}{2^B} \cdot \left[1 + \frac{N_t - 1}{N_t} \cdot 2^B \right. \\ &\quad \times \left. \left[(1-a)^{N_t} - (1-\delta)^{N_t} \right] \right] \cdot N_t \sigma_h^2 \\ &= N_t \sigma_h^2 + (N_t - 1) \cdot 2^B \left[(1-a)^{N_t} \right. \\ &\quad \left. - (2^{-B} + (1-a)^{N_t - 1})^{\frac{N_t}{N_t - 1}} \right] \sigma_h^2. \qquad (37)\end{aligned}$$

Combining (29) and (37) into (25), we obtain the following proposition.

Proposition 5.1: For i.i.d. MISO Raleigh fading channels, the average degradation of the receive SNR, for an N_t-antenna transmit beamforming system with an $N_v = 2^B$-size VQ-UEP codebook, can be approximated as

$$\begin{aligned}D_v(B) \simeq (N_t - 1) \cdot 2^B &\left[\left(2^{-B} + (1-a)^{N_t - 1} \right)^{\frac{N_t}{N_t - 1}} \right. \\ &\left. - (1-a)^{N_t} \right] \sigma_h^2 - N_t (1-a) \sigma_h^2. \quad (38)\end{aligned}$$

The average degradation of the receive SNR in (38) can be proven to be monotonically decreasing with respect to nonnegative real number B (see Appendix). Given a degradation amount D_0, this proposition provides a guideline to determine the necessary number of feedback bits. That is, we can always find the optimum integer number of feedback bits B (via, e.g., the Newton's method) with the average degradation $D_v(B)$ of the receive SNR being less than or equal to D_0. Similarly, the average receive SNR in (37) can be shown to be monotonically increasing with respect to B, and we can determine the needed number of feedback bits with the average receive SNR being less or equal to a desired γ_0^*.

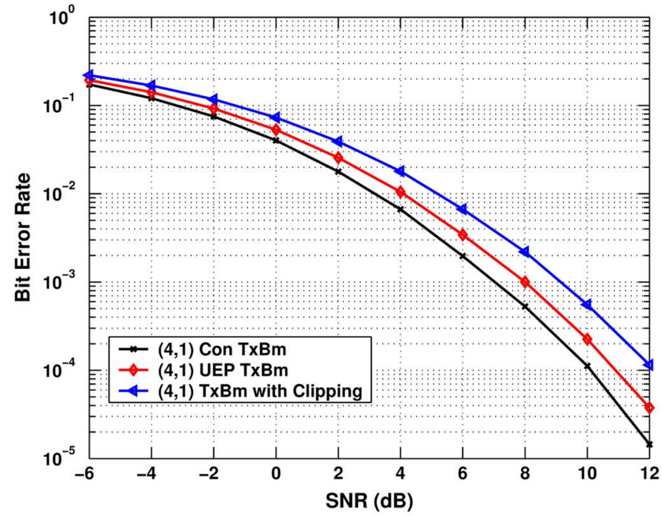
Although our analysis shares some similar features to those in [7] and [8], our results are more accurate (see Section VI). In [7] and [8], the authors found the pdf of $\xi_i = 1 - (|\mathbf{h}\hat{\mathbf{w}}_i|^2 / |\mathbf{h}\mathbf{w}_t^\circ|^2)$ via making more approximations. Under high-resolution approximations, the average degradation of the receive SNR given in [7], [8] has the form:

$$\begin{aligned}\bar{D}_v(B) &\simeq E\{|\mathbf{h}\mathbf{w}_t^\circ|^2\} E\{\xi_i\} \\ &= \left[1 + \frac{\pi}{4}(N_t - 1) \right] \sigma_h^2 \cdot 2 \left(\frac{N_t - 1}{N_t + 1} \right) 2^{-\frac{2B}{Nt-1}}. \qquad (39)\end{aligned}$$

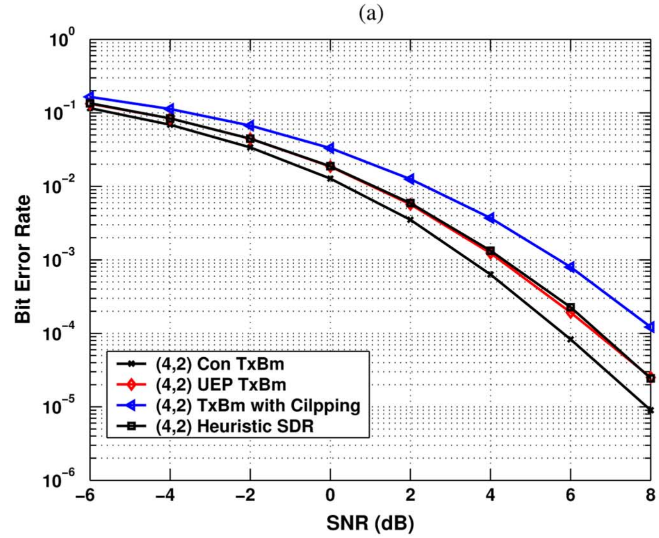
Both (38) and (39) are compared with numerically determined average receive SNR loss at the end of the next section and (38) is shown to be more accurate than (39).

VI. Numerical Examples

We present below several numerical examples to demonstrate the performance of the proposed MISO and MIMO transmit beamformer designs under the uniform elemental power constraint. We assume a frequency flat Rayleigh channel model with $E\{|h_{ij}|^2\} = 1, i = 1, 2, \ldots, N_r, j = 1, 2, \ldots, N_t$. In the simulations, we use QPSK for the transmitted symbols.



(a)



(b)

Fig. 2. Performance comparison of various transmit beamformer designs with perfect CSI at the transmitter. (a) (4,1) MISO case. (b) (4,2) MIMO case. Note that the (4,2) UEP TxBm and (4,2) Heuristic SDR curves almost coincide with each other in (b).

First, we consider the bit-error-rate (BER) performance of our proposed MISO and MIMO transmit beamformer with perfect CSI available at the transmitter. For comparison purposes, we also implement several other designs. The "Con TxBm" denotes the conventional transmit beamforming design without the uniform elemental power constraint. The "TxBm with Clipping" stands for the conventional design with peak power clipping, which means that for every transmit antenna, if $|w_{t,i}|^2 \geq (1)/(N_t)$, $w_{t,i}$ will be clipped by $w_{t,i} = w_{t,i}/(\sqrt{N_t}|w_{t,i}|), i = 1, 2, \ldots, N_t$. The "Heuristic SDR" refers to the Heuristic SDR solution described in Section III.A. We denote "UEP TxBm" as the closed-form MISO and the cyclic MIMO transmit beamformer designs under uniform elemental power constraint.

Fig. 2 shows the bit-error-rate (BER) performance comparison of various transmit beamforming designs for both the (4,1)

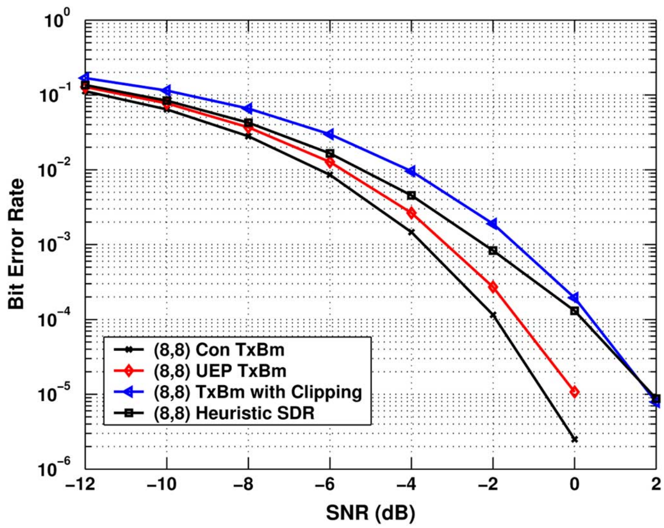


Fig. 3. Performance comparison of various transmit beamformer designs for the (8,8) MIMO case.

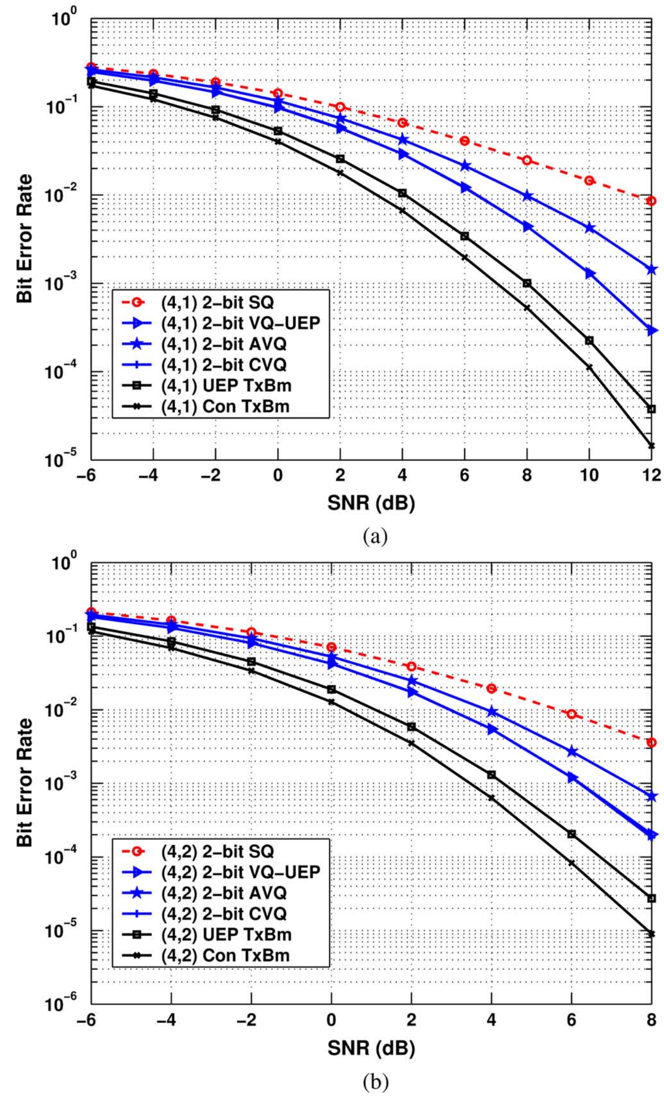


Fig. 4. Performance comparison of various transmit beamformer designs with 2-bit feedback. (a) (4,1) MISO case. (b) (4,2) MIMO case. Note that 2-bit CVQ and 2-bit VQ-UEP curves basically coincide with each other for both (4,1) and (4,2) systems, although the former is not under the uniform elemental power constraint while the latter is.

MISO and (4,2) MIMO systems. The "Con TxBm" achieves the best performance since it is not under the uniform elemental power constraint. Under the uniform elemental power constraint, the "UEP TxBm" schemes have much better performance than the "TxBm with Clipping." At $\mathrm{BER} = 10^{-3}$, for example, the improvement is about 1.5 dB for the (4,2) MIMO system. In the MIMO system, we note that our "UEP TxBm" achieves almost the same performance as the "Heuristic SDR." Interestingly, if we increase both the transmit and receive antennas to 8, as shown in Fig. 3, our "UEP TxBm" outperforms the "Heuristic SDR." The performance degradation of "Heuristic SDR" is caused by reducing the high rank optimal solution to (8) to a rank-one solution heuristically. We note here that our "UEP TxBm" is also much simpler than the "Heuristic SDR" (see the discussions in Section III).

We examine next the effects of the two quantization methods (SQ and VQ) on the overall system performance. We use herein the suboptimal combination of $\{B_i\}_{i=1}^{N_t-1}$ described in Section IV-A for SQ due to its simplicity (although the optimal one can provide a better performance). We show in Figs. 4–7 the BER performance of various quantization schemes for our proposed and conventional transmit beamformer designs, with various numbers of feedback bits ($B = 2, 4, 6, 8$). We note that VQ-UEP outperforms the AVQ for all cases. When the number of feedback bits is small (e.g., $B = 2, 4$), VQ-UEP can provide a similar performance as that of CVQ, even though the latter is not under the uniform elemental power constraint! The VQ-UEP performance approaches that of the perfect channel feedback for "UEP TxBm" when the number of feedback bits becomes larger (e.g., $B = 8$). However, CVQ needs more bits to approach the performance of its perfect channel feedback counterpart. By using relatively large numbers of feedback bits (e.g., $B = 6, 8$), we can reduce the gap between the suboptimal SQ method and VQ-UEP, since we have already reduced the number of parameters to be quantized for the scalar method due to imposing the uniform elemental power constraint.

Moreover, Fig. 8 shows the BER performance of various (2,1) MISO systems. In this case, we know that the "Alamouti Code" [1] has full rate and satisfies the uniform elemental power constraint. Compared to the "Alamouti Code," our proposed transmit beamformer design can achieve more than 2 dB SNR improvement using only a 2-bit feedback, via either the suboptimal SQ or VQ-UEP. Our proposed transmit beamformer design with a 2-bit feedback also performs similarly to its CVQ counterpart.

Finally, we examine the accuracy of the approximate degradation $D_v(B)$ of the receive SNR given in (38) for the MISO case. We carry out Monte Carlo simulations for a (4,1) system and plot the numerically simulated degradation results in Fig. 9. The training sequence size is set to $\hat{N} = 2^{17}$, and the channel variance is $\sigma_h^2 = 1$. We observe that the approximate degradation given in (38) is very close to the numerically simulated one for any feedback bit number (or rate)

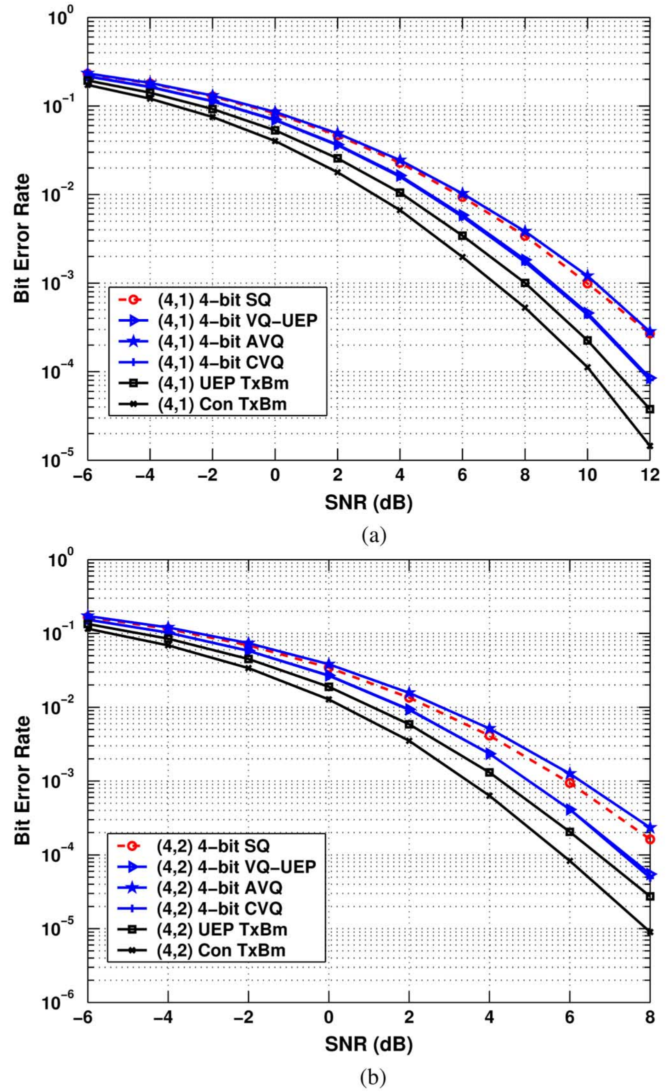


Fig. 5. Performance comparison of various transmit beamformer designs with 4-bit feedback. (a) (4,1) MISO case. (b) (4,2) MIMO case. Note that 4-bit CVQ and 4-bit VQ-UEP curves basically coincide with each other for both (4,1) and (4,2) systems, although the former is not under the uniform elemental power constraint while the latter is.

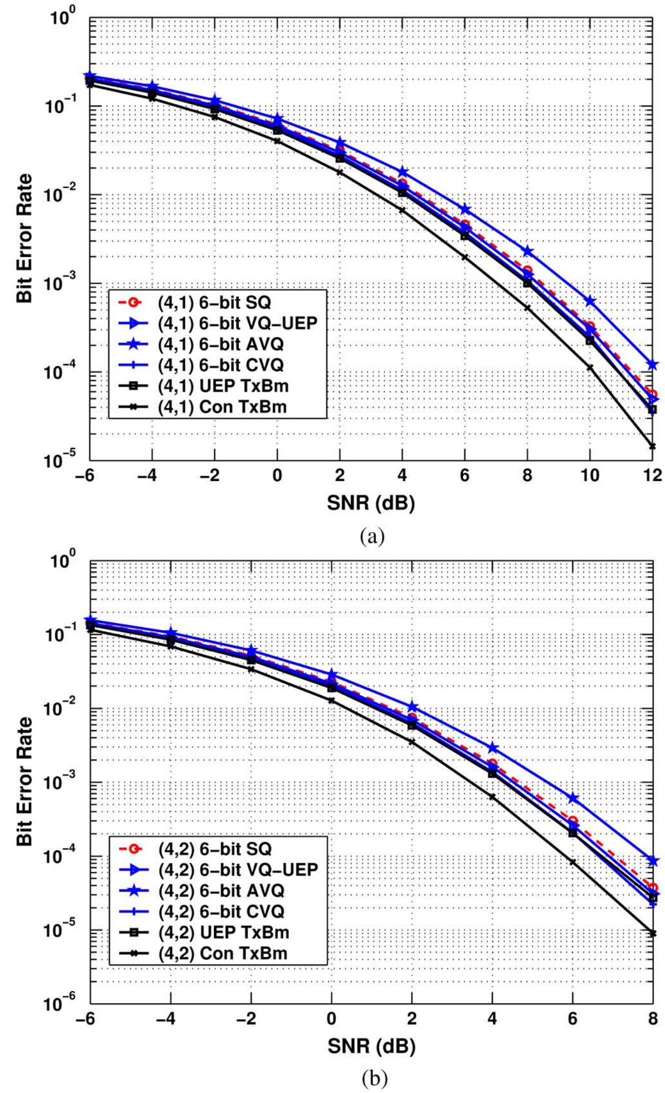


Fig. 6. Performance comparison of various transmit beamformer designs with 6-bit feedback. (a) (4,1) MISO case. (b) (4,2) MIMO case. Note that 6-bit CVQ and UEP TxBm with perfect feedback curves almost coincide with each other for both (4,1) and (4,2) systems.

B. However, the high-resolution approximation given in (39) has accurate prediction only at high feedback bit rates. Note also that the SQ and VQ-UEP perform similarly when the feedback bit number is relatively large, which means that our approximate degradation expression of the receive SNR given in (38), which is obtained for VQ-UEP, can also be used for SQ for large B.

VII. Conclusion

We have investigated MIMO transmit beamformer designs under the uniform elemental power constraint. The original problem is a difficult-to-solve nonconvex optimization problem, which can be relaxed to an easy-to-solve convex optimization problem via SDR. However, the rank reduction from an optimal SDR solution to a rank-one transmit beamfomer may degrade the system performance. We have shown that a closed-form expression for the optimal MISO transmit beamformer design exists. Then we have proposed a cyclic algorithm for the MIMO case which uses the closed-form MISO solution iteratively. This cyclic algorithm has a very low computational complexity. The numerical examples have been used to demonstrate that our proposed transmit beamformer designs outperform the conventional counterpart with peak power clipping. They can have a better performance than the Heuristic SDR solution as well.

Furthermore, we have considered finite-rate feedback techniques for our proposed transmit beamformer designs. A scalar quantization method has been proposed and shown to be quite effective when the number of feedback bits is relatively large [e.g., $B = 6, 8$ for a (4,1) or (4,2) system]. We have also proposed a vector quantization approach referred to as VQ-UEP. When the number of feedback bits is small, VQ-UEP can provide the same performance as CVQ even though the latter is not subject to the uniform elemental power

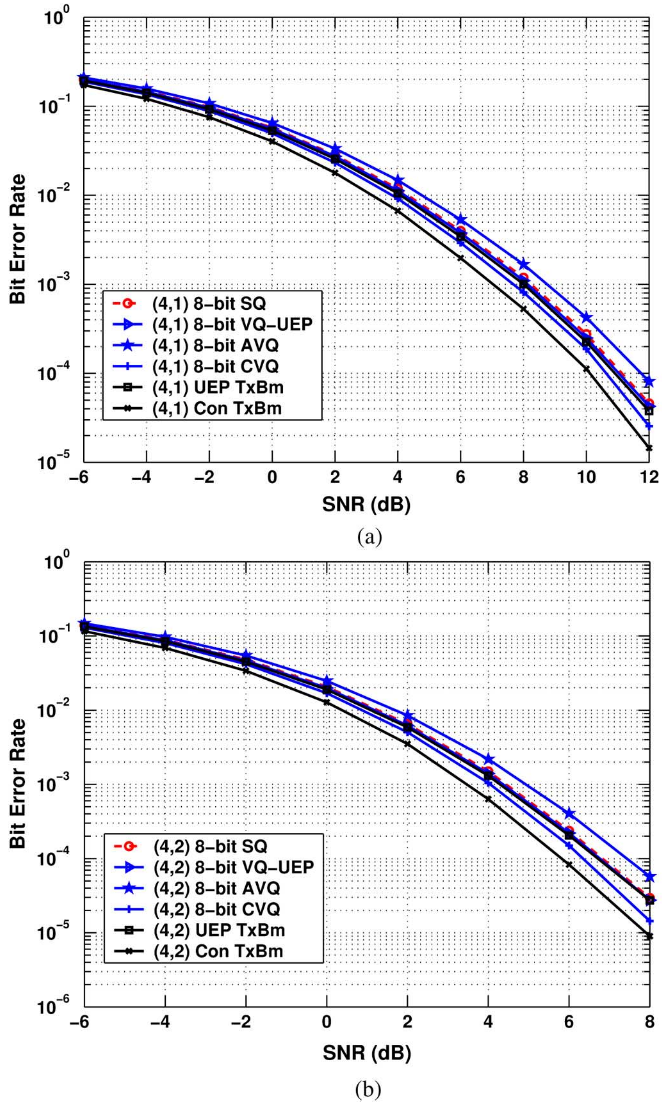


Fig. 7. Performance comparison of various transmit beamformer designs with 8-bit feedback. (a) (4,1) MISO case. (b) (4,2) MIMO case. Note that 8-bit SQ, 8-bit VQ-UEP and UEP TxBm with perfect feedback curves almost coincide with each other for both (4,1) and (4,2) systems.

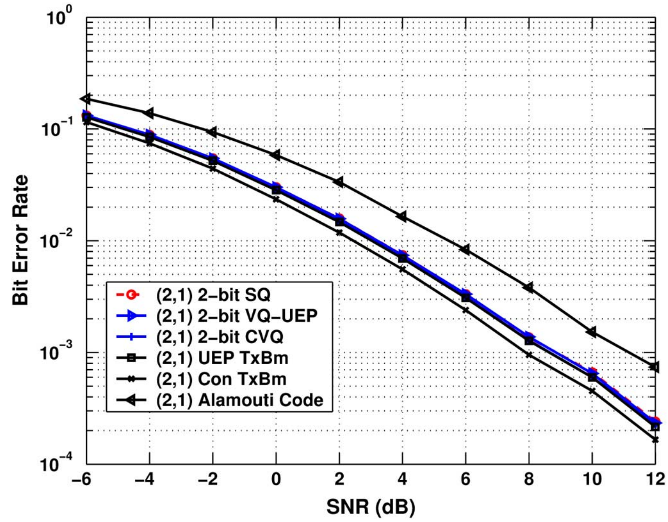


Fig. 8. Performance comparison of various (2,1) MISO systems. Note that CVQ, SQ, VQ-UEP and UEP TxBm with perfect feedback curves almost coincide with each other, although SQ and VQ-UEP are subject to both the uniform elemental power and 2-bit feedback rate constraints, while UEP TxBm assumes perfect feedback and CVQ is not subject to the uniform elemental power constraint.

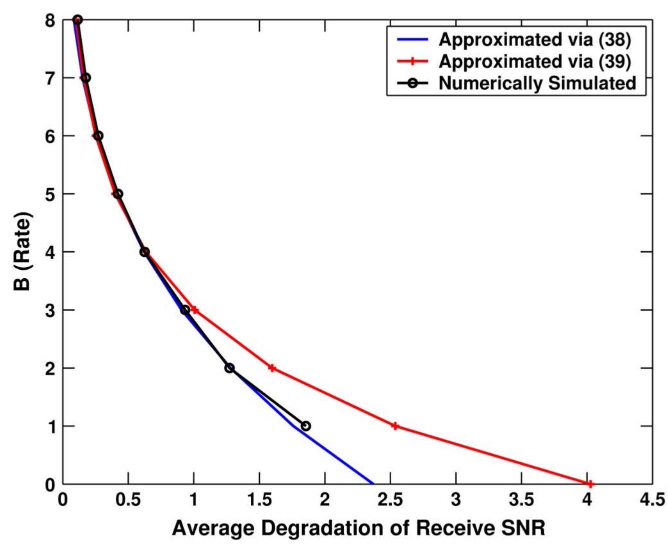


Fig. 9. Average degradation of the receive SNR for a (4,1) MISO system.

constraint. Interestingly, for a (2,1) system, our finite-rate feedback schemes can achieve more than 2 dB in SNR improvement compared to the "Alamouti Code" at the cost of requiring only a 2-bit feedback.

Finally, we have studied the average degradation of the receive SNR caused by VQ-UEP for the MISO case and obtained an approximate closed-form expression. This approximation has been shown to be quite accurate, and can serve as an accurate guideline to determine the number of feedback bits needed in a practical system.

We remark in passing that MIMO transmit beamforming has exhibited great potential for reliable wireless communications and most likely will be adopted into the next-generation wireless local area network (WLAN) standards. Although our discussions here focus on the frequency flat Rayleigh fading channels, our MIMO transmit beamformer designs can be readily extended to the frequency selective fading channels and used in, for example, MIMO-OFDM based WLAN systems.

Appendix

We prove that the average degradation $D_v(B)$ of the receive SNR in (38) is a monotonically decreasing function of the nonnegative real number B. We let $b = 2^{-B} + (1-a)^{N_t-1}$. Then the first derivative of $D_v(B)$ with respect to B is

$$\begin{aligned} D_v'(B) &= (N_t-1)\cdot \ln 2 \cdot 2^B \\ &\quad \times \left[b^{\frac{N_t}{N_t-1}} - (1-a)^{N_t} \right] \sigma_h^2 \\ &\quad + (N_t-1)\cdot 2^B \\ &\quad \times \left[\frac{N_t}{N_t-1} b^{\frac{1}{N_t-1}} (-\ln 2)\cdot 2^{-B} \right] \sigma_h^2 \\ &= c \left[2^B (1-a)^{N_t-1} \cdot b^{\frac{1}{N_t-1}} \right. \\ &\quad \left. - \frac{1}{N_t-1} b^{\frac{1}{N_t-1}} - 2^B (1-a)^{N_t} \right] \end{aligned} \tag{40}$$

where $c \triangleq (N_t-1)\cdot \ln 2 \cdot \sigma_h^2$ is a constant.

$$b^{\frac{1}{N_t-1}} < \begin{cases} (1-a)\left[1+\frac{1}{N_t-1}\cdot 2^{-B}\cdot(1-a)^{-(N_t-1)}\right], & 2^{-B} < (1-a)^{N_t-1} \\ 2^{-\frac{B}{N_t-1}}\left[1+\frac{1}{N_t-1}\cdot 2^{B}\cdot(1-a)^{N_t-1}\right], & 2^{-B} \geq (1-a)^{N_t-1}. \end{cases} \quad (44)$$

Note that

$$b^{\frac{1}{N_t-1}} = (1-a)\left[1+2^{-B}\cdot(1-a)^{-(N_t-1)}\right]^{\frac{1}{N_t-1}}. \quad (41)$$

Using the Taylor series expansion, we can expand $b^{(1)/(N_t-1)}$ as

$$b^{\frac{1}{N_t-1}} = (1-a)\sum_{n=0}^{\infty}\left[2^{-B}\cdot(1-a)^{-(N_t-1)}\right]^n \frac{f^{(n)}(1)}{n!} \quad (42)$$

or

$$b^{\frac{1}{N_t-1}} = 2^{-\frac{B}{N_t-1}}\sum_{n=0}^{\infty}\left[2^{B}\cdot(1-a)^{N_t-1}\right]^n \frac{f^{(n)}(1)}{n!} \quad (43)$$

where

$$f^{(n)}(1) = \begin{cases} 1, & n=0 \\ (-1)^n\cdot\left(-\frac{1}{N_t-1}\right)\cdots\left(n-1-\frac{1}{N_t-1}\right), & n\geq 1. \end{cases}$$

Since $(|f^{(n)}(1)|)/(n!) > (|f^{(n+1)}(1)|)/((n+1)!)$, we obtain the inequalities as shown in (44) at the top of the page.

For the $2^{-B} < (1-a)^{N_t-1}$ case, we have

$$\begin{aligned} D'_v(B) &< c\left[2^B(1-a)^{N_t-1} - \frac{1}{N_t-1}\right]\cdot(1-a) \\ &\quad \cdot\left[1+\frac{1}{N_t-1}\cdot 2^{-B}\cdot(1-a)^{-(N_t-1)}\right] \\ &\quad - c\cdot 2^B(1-a)^{N_t} \\ &= -c\left[\frac{1}{(N_t-1)^2}\cdot 2^{-B}\cdot(1-a)^{-(N_t-2)}\right] < 0. \end{aligned} \quad (45)$$

For the $2^{-B} \geq (1-a)^{N_t-1}$ case, we have

$$\begin{aligned} D'_v(B) &< c\cdot 2^{-\frac{B}{N_t-1}}\left[2^B(1-a)^{N_t-1} - \frac{1}{N_t-1}\right] \\ &\quad \times\left[1+\frac{1}{N_t-1}\cdot 2^B(1-a)^{N_t-1}\right] \\ &\quad - c\cdot 2^B(1-a)^{N_t} \\ &< \frac{c}{N_t-1}\cdot 2^{-\frac{B}{N_t-1}}\Big[2^{2B}(1-a)^{2(N_t-1)} \\ &\quad + (N_t-1)\left(1-2^{\frac{B}{N_t-1}}(1-a)\right) \\ &\quad \cdot 2^B(1-a)^{N_t-1} - 1\Big] \\ &\leq \frac{c}{N_t-1}\cdot 2^{-\frac{B}{N_t-1}} \\ &\quad \times[1+(N_t-1)(1-1)\cdot 1 - 1] = 0. \end{aligned} \quad (46)$$

Summarizing the above inequalities, we get $D'_v(B) < 0$. Thus, the average degradation of the receive SNR $D_v(B)$ is a monotonically decreasing function of the nonnegative real number B.

REFERENCES

[1] S. M. Alamouti, "A simple transmit diversity techniques for wireless communications," *IEEE J. Sel. Areas Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.

[2] V. Tarokh, H. Jafarkhani, and A. R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inf. Theory*, vol. 45, pp. 1456–1467, July 1999.

[3] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*. Cambridge, U.K.: Cambridge University Press, 2005.

[4] A. J. Grant, "Performance analysis of transmit beamforming," *IEEE Trans. Commun.*, vol. 53, pp. 738–744, Apr. 2005.

[5] S. L. Miller and R. J. O'Dea, "Peak power and bandwidth efficient linear modulation," *IEEE Trans. Commun.*, vol. 46, pp. 1639–1648, Dec. 1998.

[6] D. J. Love and R. W. Heath, Jr., "Equal gain transmission in multiple-input multiple-output wireless systems," *IEEE Trans. Commun.*, vol. 51, pp. 1102–1110, Jul. 2003.

[7] C. R. Murthy and B. D. Rao, "A vector quantization based approach for equal gain transmission," in *IEEE Global Telecommun. Conf. (GLOBECOM)*, Nov. 28–Dec. 2 2005, vol. 5, pp. 2528–2533.

[8] C. R. Murthy and B. D. Rao, "Quantization methods for equal gain transmission with finite rate feedback," *IEEE Trans. Signal Process.*, to be published.

[9] W. Yu and T. Lan, "Transmitter optimization for the multi-antenna downlink with per-antenna power constraints," *IEEE Trans. Signal Process.*, to be published.

[10] A. Narula, M. J. Lopez, M. D. Trott, and G. W. Wornell, "Efficient use of side information in multiple-antenna data transmission over fading channels," *IEEE J. Sel. Areas Commun.*, vol. 16, pp. 1423–1436, Oct. 1998.

[11] D. J. Love, R. W. Heath, Jr., and T. Strohmer, "Grassmannian beamforming for multiple-input multiple-output multiple-output wireless systems," *IEEE Trans. Inf. Theory*, vol. 49, pp. 2735–2747, Oct. 2003.

[12] K. K. Mukkavilli, A. Sabharwal, E. Erkip, and B. Aazhang, "On beamforming with finite rate feedback in multiple-antenna systems," *IEEE Trans. Inf. Theory*, vol. 49, pp. 2562–2579, Oct. 2003.

[13] S. Zhou, Z. Wang, and G. B. Giannakis, "Quantifying the power loss when transmit beamforming relies on finite-rate feedback," *IEEE Trans. Wireless Commun.*, vol. 4, pp. 1948–1957, Jul. 2005.

[14] P. Xia and G. B. Giannakis, "Design and analysis of transmit-beamforming based on limited-rate feedback," *IEEE Trans. Signal Process.*, vol. 54, pp. 1853–1863, May 2006.

[15] J. C. Roh and B. D. Rao, "Transmit beamforming in multiple-antenna systems with finite rate feedback: A VQ-based approach," *IEEE Trans. Inf. Theory*, vol. 52, pp. 1101–1112, Mar. 2006.

[16] Y. Linde, A. Buzo, and R. M. Gray, "An algorithm for vector quantizer design," *IEEE Trans. Commun.*, vol. 28, pp. 84–95, Jan. 1980.

[17] R. M. Gray and D. L. Neuhoff, "Quantization," *IEEE Trans. Inf. Theory*, vol. 44, pp. 2325–2383, Oct. 1998.

[18] J. Choi and R. W. Heath, Jr., "Interpolation based transmit beamforming for MIMO-OFDM with limited feedback," *IEEE Trans. Signal Process.*, vol. 53, pp. 4125–4135, Nov. 2005.

[19] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software Online* vol. 11–12, pp. 625–653, Oct. 1999 [Online]. Available: http://www2.unimaas.nl/sturm/software/sedumi.html

[20] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[21] R. Horst, P. M. Pardalos, and N. V. Thoai, *Introduction to Global Optimization*, 2nd ed. Boston, MA: Kluwer Academic, 2000.

[22] L. Vandenberghe and S. Boyd, "Semidefinite programming," *SIAM Rev.*, vol. 38, pp. 49–95, Mar. 1996.

[23] M. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret, "Applications of second-order cone programming," *Linear Algebra and Its Applicat., Special Issue on Linear Algebra in Control, Signals and Image Process.*, pp. 193–228, Nov. 1998.

[24] R. Orsi, U. Helmke, and J. B. Moore, “A Newton-like method for solving rank constrained linear matrix inequalities,” in *Proc. 43rd IEEE Conf. Decision Control*, 2004, pp. 3138–3144.

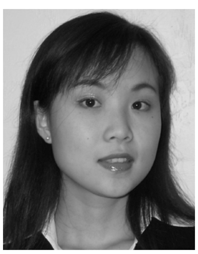
[25] P. Stoica and Y. Selen, “Cyclic minimizers, majorization techniques, and expectation-maximization algorithm: A refresher,” *IEEE Signal Process. Mag.*, pp. 112–114, Jan. 2004.

[26] G. Casella and R. L. Berger, *Statistical Inference*. Pacific Grove, CA: Duxbury, 2001.

[27] S. M. Kay, *Fundamentals of Statistical Signal Processing, Vol II - Detection Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1998.

Xiayu Zheng (S’03) received the B.Sc. and M. Sc. degrees in electrical engineering and information science from the University of Science and Technology of China (USTC), Hefei, in 2001 and 2004, respectively.

He is currently pursuing the Ph.D. degree with the Department of Electrical and Computer Engineering, University of Florida, Gainesville. His research interests are in the areas of wireless communication, signal processing, and related fields.



Yao Xie (S’05) received the B.Sc. degree from the University of Science and Technology of China (USTC), Hefei, in 2004, and the M.Sc. degree from the University of Florida, Gainesville, in 2006, both in electrical engineering.

She is currently pursuing the Ph.D. degree with the Department of Electrical Engineering at Stanford University, Stanford, CA. Her research interests include signal processing, medical imaging, and optimization.

Ms. Xie is a member of Tau Beta Pi and Etta Kappa Nu. She was the first place winner in the Student Best Paper Contest at the 2005 Annual Asilomar Conference on Signals, Systems, and Computers, for her work on breast cancer detection.

Jian Li (F’05) received the M.Sc. and Ph.D. degrees in electrical engineering from The Ohio State University, Columbus, in 1987 and 1991, respectively.

From July 1991 to June 1993, she was an Assistant Professor with the Department of Electrical Engineering, University of Kentucky, Lexington. Since August 1993, she has been with the Department of Electrical and Computer Engineering, University of Florida, Gainesville, where she is currently a Professor. Her current research interests include spectral estimation, statistical and array signal processing, and their applications.

Dr. Li is a Fellow of the Institute of Electrical Engineers (IEE). She received the 1994 National Science Foundation Young Investigator Award and the 1996 Office of Naval Research Young Investigator Award. She has been a member of the Editorial Board of Signal Processing, a publication of the European Association for Signal Processing (EURASIP), since 2005. She is presently a member of two of the IEEE Signal Processing Society technical committees: the Signal Processing Theory and Methods (SPTM) Technical Committee and the Sensor Array and Multichannel (SAM) Technical Committee.

Petre Stoica (F’94) received the D.Sc. degree in automatic control from the Polytechnic Institute of Bucharest (BPI), Bucharest, Romania, in 1979 and an honorary doctorate degree in science from Uppsala University (UU), Uppsala, Sweden, in 1993.

He is a professor of systems modeling with the Division of Systems and Control, Department of Information Technology, UU. Previously, was a Professor of system identification and signal processing with the Faculty of Automatic Control and Computers, BPI. He held visiting positions with Eindhoven University of Technology, Eindhoven, The Netherlands; Chalmers University of Technology, Gothenburg, Sweden (where he held a Jubilee Visiting Professorship); UU; The University of Florida, Gainesville; and Stanford University, Stanford, CA. His main scientific interests are in the areas of system identification, time series analysis and prediction, statistical signal and array processing, spectral analysis, wireless communications, and radar signal processing. He has published nine books, 10 book chapters, and some 500 papers in archival journals and conference records. His most recent book, coauthored with R. Moses, is *Spectral Analysis of Signals* (Englewood Cliffs, NJ: Prentice-Hall, 2005).

Dr. Stoica is on the editorial boards of six journals: *Journal of Forecasting*, *Signal Processing*, *Circuits, Signals, and Signal Processing*, *Digital Signal Processing CA Review Journal*, *Signal Processing Magazine*, and *Multidimensional Systems and Signal Processing*. He was a co-guest editor for several special issues on system identification, signal processing, spectral analysis, and radar for some of the aforementioned journals, as well as for the IEE PROCEEDINGS. He was corecipient of the IEEE ASSP Senior Award for a paper on statistical aspects of array signal processing. He was also recipient of the Technical Achievement Award of the IEEE Signal Processing Society. In 1998, he was the recipient of a Senior Individual Grant Award of the Swedish Foundation for Strategic Research. He was also a corecipient of the 1998 EURASIP Best Paper Award for Signal Processing for a work on parameter estimation of exponential signals with time-varying amplitude, a 1999 IEEE Signal Processing Society Best Paper Award for a paper on parameter and rank estimation of reduced-rank regression, a 2000 IEEE Third Millennium Medal, and the 2000 W. R. G. Baker Prize Paper Award for a paper on maximum likelihood methods for radar. He was a member of the international program committees of many topical conferences. From 1981 to 1986, he was a Director of the International Time-Series Analysis and Forecasting Society, and he was also a member of the IFAC Technical Committee on Modeling, Identification, and Signal Processing. He is also a member of the Royal Swedish Academy of Engineering Sciences, an honorary member of the Romanian Academy, and a fellow of the Royal Statistical Society.