On the Optimality of Randomized Deadlock Avoidance Policies

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Abstract

This paper revisits the problem of selecting an optimal deadlock resolution strategy, when the selection criterion is the maximization of the system throughput, and the system is Markovian in terms of its timing and routing characteristics. This problem was recently addressed in some of our previous work, that (i) provided an analytical formulation for it, (ii) introduced the notion of *randomized deadlock avoidance* as a generalization of the more traditional approaches of deadlock prevention/avoidance, and detection and recovery, and (iii) provided a methodology for selecting the optimal randomized deadlock avoidance policy for a given resource allocation system (RAS) configuration. An issue that remained open in the problem treatment of that past work, was whether the proposed policy randomization is *essential*, i.e., whether there exist any RAS configurations for which a randomized deadlock avoidance policy is superior to any other policy that does not employ randomization. The work presented in this paper establishes that for the basic problem formulation where the only concern is the (unconstrained) maximization of the system throughput – or the other typical performance objectives of minimizing the system work-in-process and mean sojourn time – randomization of the deadlock resolution strategy is *not* essential. However, it is also shown that, sometimes, it can offer an effective mechanism for accommodating additional operational constraints, like the requirement for production according to a specified product mix. Furthermore, the undertaken analysis provides an analytical characterization of the dependence of the aforementioned performance measures on the transition rates relating to the various events of the underlying state space, which can be useful for the broader problem of synthesizing efficient scheduling policies for the considered class of resource allocation systems.

Keywords: Sequential Resource Allocation Systems, Deadlock Resolution, Controlled Markov Chains, Randomized Control Policies

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1 Introduction

The work presented in this paper provides a rigorous analytical treatment and a definite response to a question raised in a prior work of ours [4, 5] regarding the extent to which randomization of a deadlock avoidance policy can provide an effective mechanism for enhancing the long-term throughput of the underlying resource allocation system (RAS). More specifically, motivated by (i) the apparent lack of a systematic analytical investigation of the impact of the applied deadlock resolution strategy on the performance of the underlying resource allocation system, and (ii) the increasing significance of this problem in current industrial contexts like that of semiconductor manufacturing, the work of [4, 5] (i) provided an analytical formulation for it, in the case that the adopted performance objective is the maximization of the long-term throughput of the underlying RAS, (ii) introduced the notion of *randomized deadlock avoidance* as a generalization of the more traditionally applied approaches of deadlock prevention/avoidance, and detection and recovery, and (iii) provided a methodology for selecting the optimal randomized deadlock avoidance policy for a given RAS configuration. An issue that remained open in the problem treatment of that past work, was whether the proposed policy randomization is *essential*, i.e., whether there exist any RAS configurations for which a randomized deadlock avoidance policy is superior to any other policy that does not employ randomization.

The work presented herein establishes that for the basic problem formulation stated above – i.e., selecting a deadlock resolution strategy that maximizes the underlying RAS throughput – and under the additional assumptions of exponentially distributed job arrival and processing times, randomization is *not* essential, i.e., there will always exist an optimal deadlock resolution strategy under which the various system transitions from its safe to its unsafe region[1] will always remain enabled or disabled.[2] It is further shown, however, that once the aforementioned basic formulation is augmented with additional constraints, like the observation of certain ratios in the production of the various job types, then, randomization can become essential, providing an effective mechanism for the imposition of the aforementioned constraints. Both results are obtained by characterizing in detail the functional dependency of the considered performance measure on the underlying system transition probabilities and the policy control parameters, which constitutes an additional contribution of the presented work. Indeed, the development of

[1] All the relevant concepts and terminology are systematically introduced in the following section.

[2] Notice, however, that even such a solution is conceptually different from the classical approaches of deadlock avoidance, and detection and recovery, since under the former (resp., the latter) approach, *all* the problematic transitions from the safe to the unsafe region are uniformly disabled (resp., enabled).

this characterization (i) allowed the immediate extension of the aforementioned results to other interesting performance measures like the minimization of the system Work-In-Process (WIP) inventory and the average job sojourn times, and (ii) it suggests an interesting framework for addressing the broader scheduling problem for the aforementioned environments and objectives; this last issue constitutes part of ongoing work [2].

The rest of the paper is organized as follows: Section 2 provides an analytical characterization of the considered problem in its basic definition, generalizing and formalizing the formulation presented in [4]. Section 3 addresses the essentiality of the DAP randomization for the basic problem formulation, by developing and analyzing the functional dependency of the considered performance measure(s) on the underlying system transition probabilities and the policy control parameters, mentioned above. Section 4 demonstrates that once the original formulation is extended by additional constraints, the policy randomization can become essential; this effect is shown by focusing on a class of constraints requesting the observation of certain production ratios among the various job types. Finally, Section 5 draws conclusions and provides some suggestions for future extensions of the presented work.

2 Preliminaries

This section provides an analytical characterization of the considered problem of selecting optimally the deadlock resolution strategy for a given RAS, by systematizing the concepts and ideas originally developed, in a more informal manner, in [4, 5].

The flexibly automated manufacturing system as a resource allocation system (RAS) and the corresponding RAS deadlock For the purposes of deadlock-related analysis, a flexibly automated production system can be pertinently abstracted to a *Resource Allocation System (RAS)*, consisting of a set $\mathcal{R} = \{R_i,\ i = 1, \ldots, m\}$ of *resource types*, and a set $\mathcal{J} = \{JT_j,\ j = 1, \ldots, n\}$ of *job / process types*, that can be executed in the system through sequential allocation of the system resources. More specifically, each resource type R_i is further characterized by its capacity C_i, i.e., a *finite* integral number indicating the number of units of this particular resource possessed by the system. Furthermore, resources are *reusable*, i.e., their allocation and deallocation to the system processes do not alter them in any way; in that sense, they constitute a system *invariant*. Jobs are executed in the system through a series of *(processing) stages*, and therefore, each job type JT_j is defined by a stage sequence: $JT_j =< JT_{jk},\ k = 1, \ldots, l(j) >$. In addition, each job stage JT_{jk} is further characterized by a resource allocation vector $A_{jk} \in (Z^+)^m$, indicating the number of resource units from each resource type that is required for the successful execution of the stage.

In the context of flexibly automated manufacturing systems, and the underlying RAS, *deadlock* arises due to the fact that a job, having finished the execution of a certain stage JT_{jk}, releases (some of) the resources allocated to it for the support of this stage, only after it has secured – i.e., been allocated – the resources for the execution of the successive stage $JT_{j,k+1}$.[3] This *"hold while waiting"* effect, combined with the exclusive and non-preemptive allocation of the finite system resources to the running jobs, can give rise to circular-waiting patterns, in which a set of jobs is permanently blocked, since each of them, in order to proceed, requires the allocation of some resource unit(s) currently held by some other job in the set. In most manufacturing system contexts, the occurrence of a deadlock is a major disruption, since, the deadlocked jobs will not be able to advance and finish through "normal" system operation, and, while the deadlock persists, the effective utilization of the resources involved is equal to zero. Furthermore, the deadlock resolution will typically require external (human) intervention, and the transfer of unfinished jobs to temporary storage.

RAS logical/structural analysis and deadlock avoidance From an analytical / methodological standpoint, the deadlock problem is systematically addressed by modeling the behavior of the considered RAS as a *Finite State Automaton (FSA)* [1]. An *event*, $e \in E$, of this FSA, corresponds to the advancement of any job in the system by one stage / step. The RAS *state*, $s \in S$, is defined by the distribution of the currently running jobs to the various processing stages supported by the system. The automaton *state transition function*, $f : S \times E \to S$, is a formal expression of the aforementioned resource allocation mechanism: $f(s, e)$ is mapped to the resulting state s', if the job step defined by event e is *feasible* under the resource allocation described by state s; otherwise, it is mapped back to state s. The *initial* and *final* states of this automaton correspond to state s_0, denoting the state in which the system is idle and empty of any jobs, and therefore, the *language* accepted by this automaton corresponds to complete production runs. Finally, we notice that this FSA model can be expressed graphically by its *State Transition Diagram (STD)*, i.e., a graph with *nodes* corresponding to the FSA states, and *arcs* corresponding to the feasible state transitions.

In the FSA formalism, deadlocks developed in the operation of the uncontrolled system are represented by the formation of *strongly connected components* in the system reachability space, – where the latter is denoted by S_r – which, however, are not *co-accessible*, i.e., the empty

[3] A typical example for this phenomenon is the allocation of the system buffering capacity available at the various workstations and material handling units. Being a physical entity, a job must always be accommodated somewhere.

state, s_0, is not reachable from them through any sequence of feasible transitions. Hence, a *correct Deadlock Avoidance Policy (DAP)*, $\mathcal{P}$, tries to restrict the system operation to a *strongly connected component of S_r which contains the empty state s_0*. Let us denote the subspace *admissible* by DAP $\mathcal{P}$ by $S_r(\mathcal{P})$. Given a RAS configuration, an applied DAP is characterized as *optimal*, if the corresponding admissible subspace is the *maximal* strongly connected component of S_r which contains the empty state s_0. The set of states admitted by the optimal DAP, $\mathcal{P}^*$, will be characterized as (the set of) *reachable safe* states, and it will be denoted by S_{rs}. The complement of S_{rs} with respect to S_r is denoted by S_{ru}, and it constitutes the system *reachable unsafe* region; formally, $S_{ru} = S_r \backslash S_{rs}$. In the context of the considered RAS's, the optimal DAP, $\mathcal{P}^*$, is well-defined, and furthermore, it is effectively computable, even though its implementation constitutes an NP-Hard problem, in general [6]. Yet, to facilitate the subsequent discussion, in the following we shall assume that the deadlock avoidance strategy implements the *optimal* DAP, $\mathcal{P}^*$; we notice, however, that the presented methodology can be applied to the comparison of any other sub-optimal DAP, $\mathcal{P}$, to the alternative strategies of detection and recovery, and randomized deadlock avoidance, by substituting in the subsequent analysis the reachable safe subspace, S_{rs}, by the policy admissible space, $S_r(\mathcal{P})$.

Modeling the alternative deadlock resolution strategies In case that the considered production system and its underlying RAS is operated under the detection and recovery strategy, the system is allowed to access its entire reachable subspace S_r. Furthermore, whenever a deadlock is reached, the involved processes are identified, and the deadlock is resolved by *swapping* (a subset of) the deadlocked processes in a way that it will allow their further progress. This job swapping mechanism constitutes the *deadlock recovery* phase of the system operation, and, from the implementational standpoint, it could be either human-driven or totally automated. In the FSA modeling context, the swapping of (some of) the deadlocked jobs corresponds to a single transition from the deadlocked state s_d to another deadlock-free state s'. Since s' is reached through the job swapping mechanism, which constitutes an exception handling procedure, it is possible – in fact, quite typical in any actual implementation of this strategy – that $s' \in S \backslash S_r$. From s', the autonomous "normal" operation of the system is resumed, until the system reaches another deadlocked state, in which case, the deadlock detection and recovery scheme described above is repeated on this new state. Hence, in the FSA modeling framework, the deadlock detection and recovery approach establishes the ability of the system to run to completion, even under the occurrence of deadlocks, through the insertion of additional transitions to the STD modeling the original feasible system behavior, that correspond to the deadlocked process swaps by some sort of exception handling routine. In other words, the insertion of these new transitions ensures that for every RAS state $s \in S$, visited by the system when operated under the deadlock detection and recovery strategy, it holds: $s \xrightarrow{*} s_0$.

Finally, the randomized deadlock avoidance strategy operates similar to the detection and recovery approach, with the additional feature that resource allocation requests corresponding to transitions $t: \ s \to s'$, with $s \in S_{rs} \wedge s' \in S_{ru}$, are satisfied only with a certain probability ω_t. In particular, assuming that $\omega_t \neq 0, \ \forall t$, the reachable state space for a given RAS configuration under the randomized deadlock avoidance strategy is identical to the corresponding state space that is reachable when the system is operated under the detection and recovery approach.

RAS performance modeling and optimization Under the assumption that the timing of the various events identified in the STD modeling the system behavior under a given deadlock resolution strategy is *exponentially* distributed, the system *timed* dynamics can be effectively modeled by a *Continuous Time Markov Chain (CTMC)* [1]. To formalize the subsequent development, consider a given RAS configuration, controlled by a *randomized DAP (R-DAP)* $\mathcal{P}$, and let $S_r(\mathcal{P}) \subseteq S$ denote the resulting reachability space. Furthermore, for every transition $t_{ij}: \ s_i \xrightarrow{\mathcal{P}} s_j, \ s_i, s_j \in S_r(\mathcal{P})$, that is feasible under the considered policy, let $\bar{q}_{ij}$ denote the rate of the exponential distribution characterizing the *natural* timing of the corresponding event.[4] Under the control of R-DAP $\mathcal{P}$, the occurrence rate of transitions $t_{ij}: \ s_i \to s_j$, with $s_i \in S_{rs} \ \wedge \ s_j \in S_{ru}$, is moderated by the transition control probability $\omega_{ij} \in [0, 1]$ to $q_{ij} = \omega_{ij} \cdot \bar{q}_{ij}$. As a result, the *infinitesimal generator matrix* Q, defining the CTMC that describes the system dynamics when it is controlled by R-DAP $\mathcal{P}$, is given by $Q = [q_{ij}]$ with

$$q_{ij} = \begin{cases} \omega_{ij}\bar{q}_{ij} & \text{if } s_i, \ s_j \in S_r(\mathcal{P}) \ \wedge i \neq j \ \wedge \ s_i \xrightarrow{\mathcal{P}} s_j \ \wedge s_i \in S_{rs} \ \wedge \ s_j \in S_{ru} \\ \bar{q}_{ij} & \text{if } s_i, \ s_j \in S_r(\mathcal{P}) \ \wedge \ i \neq j \ \wedge \ s_i \xrightarrow{\mathcal{P}} s_j \ \wedge (s_i \notin S_{rs} \ \vee \ s_j \notin S_{ru}) \\ 0 & \text{if } \ s_j \in S_r(\mathcal{P}) \ \wedge i \neq j \ \wedge \ s_i \stackrel{\mathcal{P}}{\not\to} s_j \\ -\sum_{j: \ j \neq i} q_{ij} & \text{if } s_i \in S_r(\mathcal{P}) \ \wedge \ i = j \end{cases} \quad (1)$$

In the formalism of Equation 1, the system dynamics under the control of (classical) deadlock avoidance (resp., detection and recovery) strategy, are modeled by setting $\omega_{ij} = 0$ (resp., 1) $\forall (i,j): \ s_i \to s_j$ with $s_i \in S_{rs} \wedge s_j \in S_{ru}$. Furthermore, since, under any deadlock resolution strategy, the resulting system behavior is irreducible, aperiodic, finite-state, and therefore, er-

[4] In the considered operational context of flexibly automated manufacturing systems, $\bar{q}_{ij}$ correspond to the job arrival/loading, processing and unloading rates, as well as the rates characterizing the job swapping mechanism during the deadlock recovery phase, and they are determined by factors like the adopted technology and the applied operational policies.

godic, the CTMC defined by Equation 1 has a *unique limiting stationary distribution*, expressed by the *steady state probability vector* π, obtained by the following system of equations [1]:[5]

$$\pi^T Q = \mathbf{0}^T \tag{2}$$

$$\sum_{\{i:\ s_i \in S_r(\mathcal{P})\}} \pi_i = 1.0 \tag{3}$$

Given the stability implied by the ergodic nature of the system behavior, a characterization of the steady-state (long-run) system throughput can be obtained by considering the cumulative rate according to which jobs are loaded into the system. Therefore, recognizing that the steady state probabilities π_i can be interpreted as the percentage of time that the RAS spends in each state $s_i \in S_r(\mathcal{P})$, while element q_{ij} denotes the rate according to which the system transitions from state s_i to state s_j, once in state s_i, the cumulative job loading rate expressing the system throughput, under R-DAP $\mathcal{P}$, is given by:

$$TH(\mathcal{P}) = \sum_{\{(i,j):\ \text{transition } s_i \xrightarrow{\mathcal{P}} s_j \text{ corresponds to a job loading event}\}} \pi_i q_{ij} \tag{4}$$

Finally, Equations $1-4$, combined with the fact that, in the considered modeling framework, a R-DAP($\mathcal{P}$) is essentially defined by the values assigned to the probabilities controlling the transition rates from the safe to the unsafe region of the underlying RAS – to be collectively denoted by the vector ω – imply that the *optimal deadlock resolution strategy selection problem* can be formally stated as follows:

$$\max_{\omega} TH(\omega; \bar{q}_{ij}) = \sum_{\{(i,j):\ \text{transition } s_i \xrightarrow{\mathcal{P}} s_j \text{ corresponds to a job loading event}\}} \pi_i q_{ij} \tag{5}$$

s.t.

$$\pi^T Q(\omega; \bar{q}_{ij}) = \mathbf{0}^T \tag{6}$$

$$\sum_{i:\ s_i \in S_r(\mathcal{P})} \pi_i = 1.0 \tag{7}$$

$$\forall i, j,\ \omega_{ij} \in [0, 1] \tag{8}$$

Example To provide a concrete example of the concepts introduced above, consider the small RAS depicted in Figure 1. This RAS consists of two resources, R_1 and R_2 of unit capacity, and it supports the execution of two job types, JT_1 and JT_2, with respective process plans

[5] In the following, **boldface** elements in the presented equations denote column vectors.

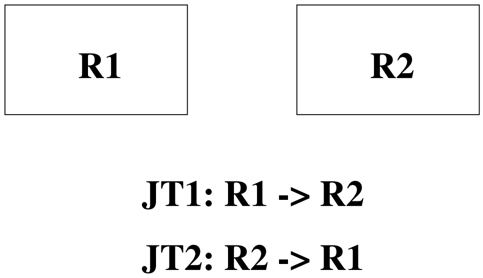

Figure 1: Example: The considered RAS

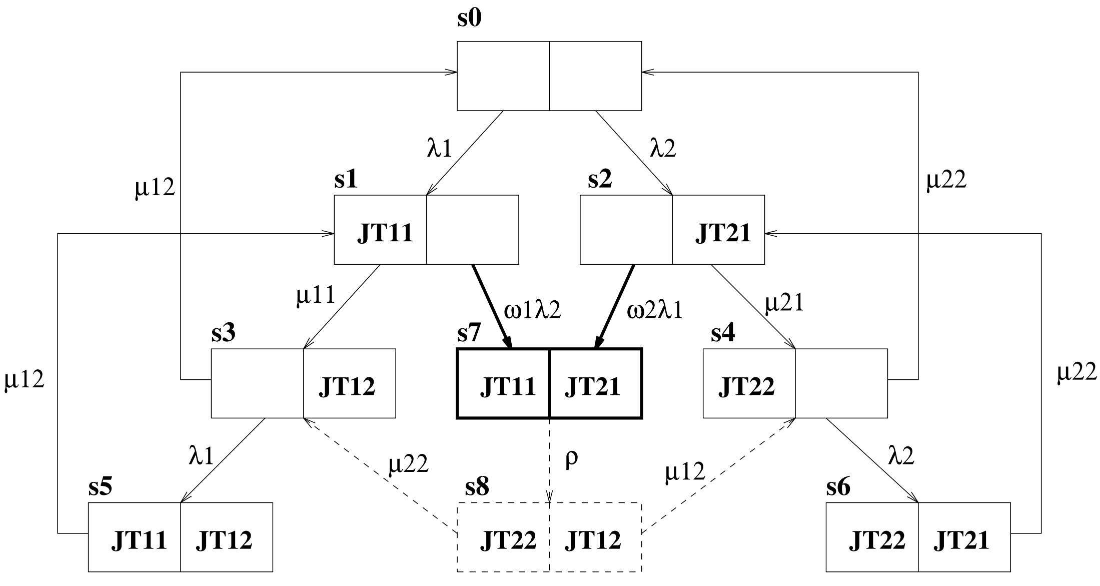

Figure 2: Example: The state space describing the system behavior under various deadlock resolution strategies

$JT_1 : < [1,0]^T, [0,1]^T >,\ JT_2 : < [0,1]^T, [1,0]^T >$.[6] The STD modeling the system state space under the control of the three deadlock resolution strategies considered in this work is depicted in Figure 2. Specifically, the uncontrolled system behavior is modeled by the subgraph induced by the state subset $S_r = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$. Furthermore, the reachable safe subspace, admitted by the optimal deadlock avoidance policy is defined by the state subset $S_{rs} = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$, while the reachable unsafe subspace consists of the singleton $S_{ru} = \{s_7\}$, which constitutes the unique deadlock state. Hence, the transitions from S_{rs} to S_{ru}, that must be controlled under randomized deadlock avoidance, are $t_1^d = s_1 \rightarrow s_7$, and $t_2^d = s_2 \rightarrow s_7$. In the following, we shall denote the respective control probabilities by ω_1 and ω_2. Moreover, under the randomized deadlock avoidance and the detection and recovery strategies, the deadlock of state s_7 is resolved by swapping the two deadlocked jobs, and the resulting state, s_8, is a state that is unreachable under the "normal" system operation. Finally, once in state s_8, the system will return to its safe region S_{rs}, either through state s_3 or through state s_4.

In order to characterize the timed dynamics of this system and its expected throughput at steady state, let us further assume that the loading time for jobs of type $JT_i,\ i = 1, 2$, follows an exponential distribution with rate $\lambda_i,\ i = 1, 2$. Similarly, the processing time for stage $JT_{ij},\ i, j = 1, 2$, is exponentially distributed with rate μ_{ij}. Finally, the time required for recovering from the deadlock of state s_7 is exponentially distributed with rate $\rho > 0$.[7] Then, the occurrence rates for the various system transitions are annotated in Figure 2, while the

[6] This RAS could model, for instance, the allocation of the buffering capacity in a two-machine robotic cell, or a two-chamber cluster tool, supporting the processing of two parts with counterflow process plans.

[7] The requirement that ρ is strictly greater than 0 enforces that the applied detection and recovery strategy as well as any R-DAP contain indeed a deadlock recovery mechanism, and they do not let the system get permanently stuck in the deadlock state.

infinitesimal matrix generator for the resulting CTMC is as follows:

$$Q = \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & & & & & & \\ & -(\mu_{11}+\omega_1\lambda_2) & & \mu_{11} & & & & \omega_1\lambda_2 & \\ & & -(\mu_{21}+\omega_2\lambda_1) & & \mu_{21} & & & \omega_2\lambda_1 & \\ \mu_{12} & & & -(\mu_{12}+\lambda_1) & & \lambda_1 & & & \\ \mu_{22} & & & & -(\mu_{22}+\lambda_2) & & \lambda_2 & & \\ & \mu_{12} & & & & -\mu_{12} & & & \\ & & \mu_{22} & & & & -\mu_{22} & & \\ & & & & & & & -\rho & \rho \\ & & & \mu_{22} & \mu_{12} & & & & -(\mu_{22}+\mu_{12}) \end{bmatrix} \quad (9)$$

Denoting the system steady state probabilities by $\pi_i,\ i = 0, \ldots, 8$, the cumulative job loading rate at steady state, defining its (long-run) throughput, is given by:

$$TH = \lambda_1\pi_0 + \lambda_2\pi_0 + \omega_1\lambda_2\pi_1 + \omega_2\lambda_1\pi_2 + \lambda_1\pi_3 + \lambda_2\pi_4 \quad (10)$$

Therefore, the problem of selecting the optimal deadlock resolution strategy for this example system is posed as follows:

$$\max_{\omega_1,\omega_2} TH(\omega_1, \omega_2; \lambda_i, \mu_{ij}, \rho,\ i,j = 1,2) = \lambda_1\pi_0 + \lambda_2\pi_0 + \omega_1\lambda_2\pi_1 + \omega_2\lambda_1\pi_2 + \lambda_1\pi_3 + \lambda_2\pi_4 \quad (11)$$

s.t.

$$\pi^T Q(\omega_1, \omega_2; \lambda_i, \mu_{ij}, \rho,\ i,j = 1,2) = \mathbf{0}^T \quad (12)$$

$$\sum_{i=0}^{8} \pi_i = 1.0 \quad (13)$$

$$\omega_1,\ \omega_2 \in [0,1] \quad (14)$$

□

3 Optimal Solutions and Extreme Points

In this section, it is shown that the problem of selecting the optimal deadlock resolution strategy, when formulated according to Equations 5 – 8, has always an optimal solution that is an *extreme point* – i.e., a corner point – of the hypercube $[0,1]^{\dim(\omega)}$. This result is developed in two main steps: (i) First, the functional dependence of the objective function of Equation 5 on the problem decision variables ω_{ij}, under constraints (6) and (7), is analytically characterized. (ii) Subsequently, it is established that the derived functional form, when constrained in the hypercube $[0,1]^{\dim(\omega)}$, always possesses a maximal value that is an extreme point of the considered domain.

The result relating to the first of the aforementioned steps is formally stated and proven as follows:

Proposition 1 *The optimization problem of Eqs 5 – 8 can be transformed to an equivalent optimization problem of the form:*

$$\max_{\omega} TH(\omega; \bar{q}_{ij}) = \frac{N(\omega; \bar{q}_{ij})}{D(\omega; \bar{q}_{ij})} \tag{15}$$

s.t.

$$\forall i, j, \ \omega_{ij} \in [0, 1] \tag{16}$$

where $N(\omega; \bar{q}_{ij})$ and $D(\omega; \bar{q}_{ij})$, appearing in Equation 15, are first-degree polynomials w.r.t. each of the decision variables ω_{ij}.[8]

Proof: Since the CTMC defined by matrix Q is ergodic, it possesses a unique stationary distribution, π, which can always be effectively computed by solving the following system of equations [1]:

$$\pi^T \left[\begin{array}{cc} \hat{Q}(\omega; \bar{q}_{ij}) & \mathbf{1} \end{array} \right] = \left[\begin{array}{cc} \mathbf{0^T} & 1 \end{array} \right] \tag{17}$$

In Equation 17, $\hat{Q}(\omega; \bar{q}_{ij})$ denotes the matrix obtained from the chain infinitesimal generator $Q(\omega; \bar{q}_{ij})$ by removing its first column, corresponding to state s_0.[9] To facilitate the subsequent discussion, let us rewrite Equation 17 in the most familiar form of:

$$\left[\begin{array}{c} \hat{Q}^T(\omega; \bar{q}_{ij}) \\ \mathbf{1}^T \end{array} \right] \pi = \left[\begin{array}{c} \mathbf{0} \\ 1 \end{array} \right] \tag{18}$$

Furthermore, let us denote by A the system matrix in the left-hand-side (lhs) of Equation 18. Then, regarding A, the following remarks hold: (i) A is a square invertible matrix, and therefore, $\det(A)$ will exist and it will always have a non-zero value. (ii) Since, by the problem definition, each control variable ω_{ij} is associated uniquely with an unsafe transition linking the safe to the unsafe region of the system state space, it appears in a unique row of $\hat{Q}$, and therefore, in a unique column of A. More specifically, ω_{ij} will appear in the column corresponding to the π_i component of the steady state probability vector.

[8] This type of functions are characterized as *multi-linear* in the relevant literature.

[9] Notice that, for a well-defined RAS, all transitions emanating from and/or leading to state s_0 are safe. Hence, under R-DAP control, $q_{ij} = \bar{q}_{ij}$ (c.f., Eq. 1), and therefore, by dropping this first column of Q, we still maintain all the problem control variables, ω_{ij}.

The first of the above observations implies that the system steady state probabilities, π_k, can always be computed as functions of ω_{ij} and $\bar{q}_{ij}$, by means of Cramer's rule [7]:

$$\pi_k(\omega; \bar{q}_{ij}) = \frac{\det(A_k(\omega; \bar{q}_{ij}))}{\det(A(\omega; \bar{q}_{ij}))} \tag{19}$$

where $A_k(\omega; \bar{q}_{ij})$ is the matrix obtained from $A(\omega; \bar{q}_{ij})$, by substituting its column corresponding to variable π_k with the right-hand-side (rhs) vector of Equation 18. The second of the above remarks regarding $A(\omega; \bar{q}_{ij})$, implies the following: (i) Both, the numerator, $\det(A_k(\omega; \bar{q}_{ij}))$ and the denominator, $\det(A(\omega; \bar{q}_{ij}))$, in the rhs of Equation 19, are first-degree polynomials w.r.t. to each control variable ω_{ij}. (ii) The numerator, $\det(A_k(\omega; \bar{q}_{ij}))$, is independent of the particular ω_{ij} with $i = k$ (assuming that such an ω-variable exists in the original problem definition).

The last two remarks further imply that each steady state probability π_k is functionally dependent on ω according to the fractional form specified by Proposition 1. Moreover, the second of these remarks implies that this functional form applies also to the products $\pi_k q_{kj}$, since it guarantees that in the case of controlled transitions t_{kj} corresponding to loading events, where $q_{kj} = \omega_{kj}\lambda_{kj}$, the numerator of π_k, $\det(A_k(\omega; \bar{q}_{ij}))$, is itself independent of ω_{kj}. Finally, since all products $\pi_k q_{kj}$ have the same denominator $\det(A(\omega; \bar{q}_{ij}))$, the functional form defined in Proposition 1 applies also to their summation. But then, the result of Proposition 1 is established simply by noticing that the problem objective function, defined in Equation 5, is just the summation of a pertinently selected subset of the product terms $\pi_k q_{kj}$, while Equation 16 is simply the last constraint in the original problem formulation. □

The next lemma will be used as a stepping stone in order to prove that the optimization problem defined by Eqs 15 – 16 has an optimal solution that is an extreme point of its feasible region. Its proof can be obtained through a simple monotonicity argument based on the form of the derivative of function $f(x)$, and therefore, it is omitted.

Lemma 1 *The* single-variable *optimization problem:*

$$\max_x f(x) = \frac{ax + b}{cx + d} \tag{20}$$

s.t.

$$x \in [0, 1] \tag{21}$$

with the additional assumption that

$$cx + d \neq 0, \quad \forall x \in [0, 1] \tag{22}$$

always has an optimal solution in the set $\{0, 1\}$.

Finally, the main result of this section is stated and proven in the following theorem:

Theorem 1 *The optimal deadlock resolution strategy selection problem, defined in Equations 5 – 8, always has an optimal solution that is an extreme point of the hypercube* $[0,1]^{dim(\omega)}$.

Proof: We prove this result by contradiction, utilizing the transformed problem version of Proposition 1. Hence, suppose that *all* optimal solutions to the optimization problem defined by Eqs 15 – 16 are interior points of the hypercube $[0,1]^{\mathrm{dim}(\omega)}$.[10] Let ω^* denote such an optimal interior point. Then, there must exist a component ω^*_{kl} of ω^* s.t. $\omega^*_{kl} \in (0,1)$. Consider the function $TH(\omega_{kl};\ \omega^*_{ij},\ i \neq k \vee j \neq l,\ \bar{q}_{ij})$. This function is a single-variable function possessing the fractional form defined in Lemma 1, and it is defined over the interval $[0,1]$. Yet, under the considered hypothesis, its maximal value is obtained at $\omega^*_{kl} \in (0,1)$, which contradicts the result of Lemma 1 (and establishes the truth of Theorem 1). □

Example Let us consider the example RAS introduced in Section 2, when the loading and processing rates take the following values: $\lambda_1 = 1.0$, $\lambda_2 = 1.0$, $\mu_{11} = 3.0$, $\mu_{12} = 2.0$, $\mu_{21} = 1.0$ and $\mu_{22} = 2.0$. Table 1 provides the analytical form of the system throughput function, $TH(\omega_1, \omega_2;\ \rho)$, as characterized by Equation 15, for three different values of the deadlock recovery rate, ρ. Figure 3 also provides the plots of these functions over their domain area $[0,1]^2$. As expected, in all cases, the resulting throughput function obtains its maximum (and minimum) value at one of the extreme points of its domain; the detailed characterization of the optimal solution and the maximal value of the objective function are included in Table 1. It is interesting to notice that for the case of $\rho = 0.5$, the maximal throughput is obtained for $(\omega^*_1, \omega^*_2) = (0,1)$, which is a strategy conceptually different from the two classical approaches of deadlock avoidance, and deadlock detection and recovery. However, as the deadlock recovery rate ρ increases (resp., decreases), the optimal strategy switches to deadlock detection and recovery (resp., deadlock avoidance), since the (time) cost of deadlock recovery becomes lower (resp., prohibitively higher than the productivity gains obtained from the enhanced operational concurrency). □

Generalizing Theorem 1 Concluding this section, we notice that even though the result of Theorem 1 was developed for the case that the optimized objective was the (long-run) system throughput, it can be easily generalized to all variations of the formulation of Eqs 5 – 8,

[10]Notice that (i) the functional form of $TH(\omega; \bar{q}_{ij})$, implied by Proposition 1, (ii) the fact that $D(\omega; \bar{q}_{ij}) \equiv \det(A(\omega; \bar{q}_{ij})) \neq 0$, $\forall \omega$, and (iii) the finiteness of the problem feasible region (c.f., Eq. 16), imply that the considered problem is well-defined.

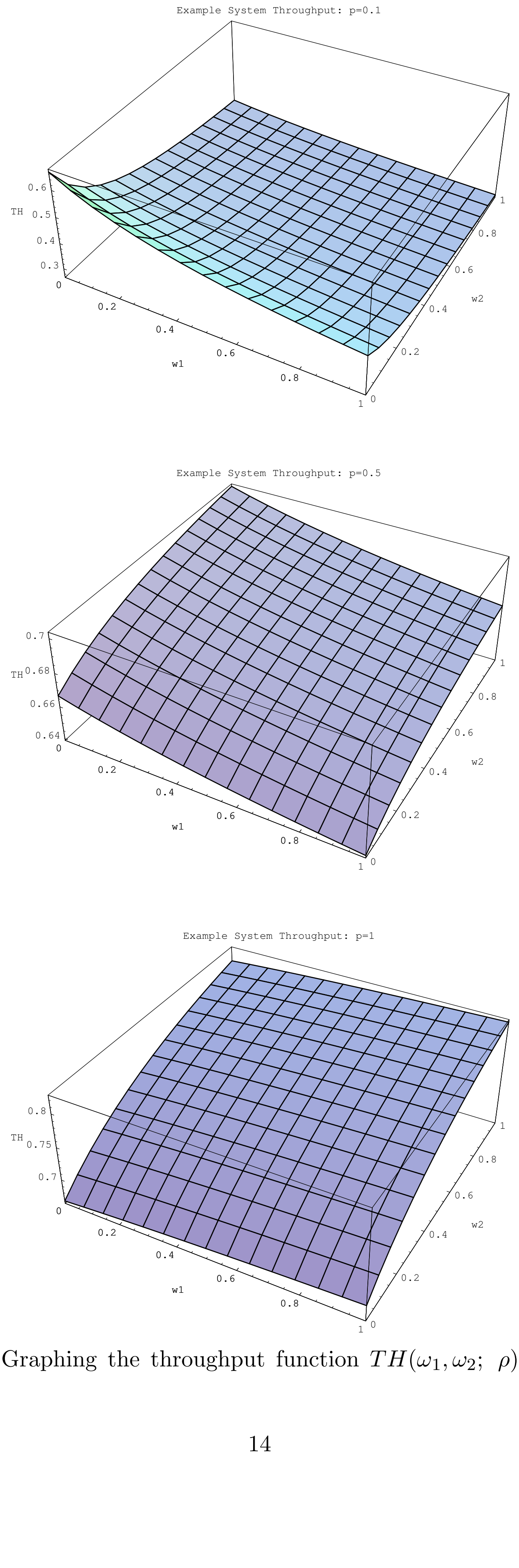

Figure 3: Example: Graphing the throughput function $TH(\omega_1, \omega_2;\ \rho)$ for the considered ρ values

ρ	$TH(\omega_1, \omega_2)$	(ω_1^*, ω_2^*)	$TH(\omega_1^*, \omega_2^*)$
0.1	$\frac{6(12+7\omega_1+21\omega_2+12\omega_1\omega_2)}{108+166\omega_1+462\omega_2+399\omega_1\omega_2}$	$(0, 0)$	0.667
0.5	$\frac{6(12+7\omega_1+21\omega_2+12\omega_1\omega_2)}{108+70\omega_1+174\omega_2+111\omega_1\omega_2}$	$(0, 1)$	0.702
1.0	$\frac{6(12+7\omega_1+21\omega_2+12\omega_1\omega_2)}{108+58\omega_1+138\omega_2+75\omega_1\omega_2}$	$(1, 1)$	0.823

Table 1: Example: The system throughput as a function of the control variables, ω_1 and ω_2, and its maximal value, for various values of the deadlock recovery rate, ρ.

where the optimized function is a linear combination of the system steady-state probabilities, π, provided that the coefficient multiplying the steady-state probability π_k is only a function of ω_{ij} with $i = k$. As a more concrete example, consider the objective of minimizing the average number of parts in the system, $\bar{N}$. It can be easily seen that, under steady-state operation, this statistic is given by:

$$\bar{N} = \sum_{i=0}^{S_r(\mathcal{P})} N(s_i)\pi_i \tag{23}$$

where $N(s_i)$ denotes the number of parts in the system when it is in state $s_i \in S_r(\mathcal{P})$. Since $N(s_i)$ is independent of ω, it follows that Equation 23 is minimized, under the constraints of Eqs 6 − 8, at one of the extreme points of the hypercube $[0, 1]^{\dim(\omega)}$. Furthermore, noticing that, (i) by Little's law [1], the average job sojourn time, $\bar{D}$, can be expressed by

$$\bar{D} = \frac{\bar{N}}{TH} \tag{24}$$

and (ii) the fractional functional forms expressing the dependency of the quantities $\bar{N}$ and TH on the control variable vector ω (c.f., Eq. 15) have the same denominator, $\det(A(\omega; \bar{q}_{ij}))$, it can be concluded that the functional dependence of the average sojourn time, $\bar{D}$, on the control vector, ω, is also expressed by a fractional form where, both, the numerator and the denominator are first-degree polynomials w.r.t. each control variable ω_{ij}. Hence, this important system performance measure, under the constraints of Eqs 6 − 8, is also optimized at an extreme point of the hypercube $[0, 1]^{\dim(\omega)}$.

4 Product-mix Considerations and Optimal R-DAP's

The formal arguments developed in the previous section in order to establish the non-essential role of randomization in the optimization of the applied deadlock resolution strategy with respect to the performance criteria of throughput, WIP inventories, and the job expected sojourn times, reveal also the key problem elements that underly this result. Specifically, Proposition 1 and Theorem 1 imply that the aforementioned result is based on (i) the ability to express the considered objective functions as a fraction of two multi-linear functions of the control variables, and (ii) the structure of the solution space, which is the entire hypercube $[0,1]^{\dim(\omega)}$. Furthermore, both of these problem properties result from the fact that the various control probabilities, ω_{ij}, are mutually independent. It follows then that the negation of this mutual independence can lead to problem variations for which the randomization of the applied deadlock resolution strategy can be essential for achieving optimum performance.

As a case in point, in this section we consider the problem variation where the formulation of Equations 5 – 8 is augmented with an additional constraint of the type:

$$\frac{TH_k}{TH_l} = \xi_{kl}, \quad (k,l) \in \mathcal{C} \subseteq \mathcal{J} \times \mathcal{J} \tag{25}$$

This constraint enforces a *"product-mix"* specification on the operation of the underlying production system, and it is a requirement that arises naturally in many multi-item production systems either due to higher-level production planning taking place in the company, or due to the fact that the considered parts constitute components for a higher-level (sub-)assembly, produced in some downstream operation of the overall supply chain [3]. In the following, first we demonstrate, through an example, how the aforementioned type of constraint establishes some coupling among the problem control variables which restrains the feasible region for the original formulation of Equations 5 – 8, and leads to a randomized optimal solution for the underlying optimization problem. Furthermore, in the second part of the section, we establish an interesting topological property for the optimal solution(s) of the extended problem formulation addressed in this section.

Example To provide a concrete example of the way that Constraint 25 affects the feasible region of the original problem formulation of Equations 5 – 8, consider the example system of Figure 1, under the parameterization introduced in the example of Section 3, for the particular case that the deadlock recovery rate $\rho = 1.0$. We remind the reader that, for this particular system, the system total throughput depends on the control variables ω_1 and ω_2 as follows:

$$TH(\omega_1,\omega_2) = \frac{6(12 + 7\omega_1 + 21\omega_2 + 12\omega_1\omega_2)}{108 + 58\omega_1 + 138\omega_2 + 75\omega_1\omega_2} \tag{26}$$

Furthermore, the partial throughput functions for each of the two job types can be obtained in a similar fashion, and they are as follows:

$$TH_1(\omega_1,\omega_2) = \frac{6(6 + 4\omega_1 + 9\omega_2 + 6\omega_1\omega_2)}{108 + 58\omega_1 + 138\omega_2 + 75\omega_1\omega_2} \tag{27}$$

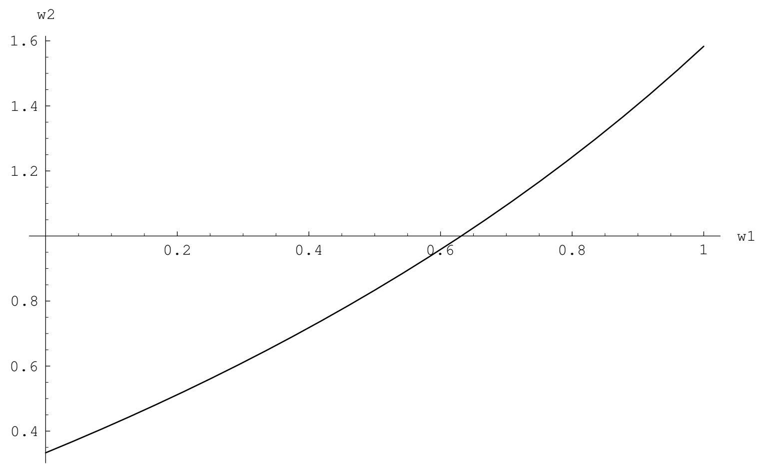

Figure 4: Example: Graphing the problem feasible region under the additional constraint of Equation 29, and with $\xi = 0.9$

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$$TH_2(\omega_1, \omega_2) \quad = \quad \frac{18(2 + \omega_1 + 4\omega_2 + 2\omega_1\omega_2)}{108 + 58\omega_1 + 138\omega_2 + 75\omega_1\omega_2} \tag{28}$$

Hence, in this particular case, the product-mix constraint of Equation 25 takes the form:

$$\frac{6 + 4\omega_1 + 9\omega_2 + 6\omega_1\omega_2}{3(2 + \omega_1 + 4\omega_2 + 2\omega_1\omega_2)} = \xi \tag{29}$$

For a value of $\xi = 0.9$, Equation 29 can be solved for ω_2, giving:

$$\omega_2 = -\frac{0.6 + 1.3\omega_1}{-1.8 + 0.6\omega_1} \tag{30}$$

A plot of Equation 30 for $\omega_1 \in [0, 1]$ is provided in Figure 4. Notice that Constraint 8 in the original problem formulation implies that the actual feasible region for the modified problem is the segment of the depicted curve contained among the points $(0, 1/3)$ and $(12/19, 1)$. Hence, plotting the function of Equation 26 for $\omega_1 \in [0, 12/19]$, and with ω_2 computed according to Equation 30, we obtain the curve depicted in Figure 5. From this figure, it can be deduced that, for the considered product-mix requirement, the system throughput is maximized by applying a randomized DAP with $(\omega_1, \omega_2) = (12/19, 1.0)$. $\square$

Some interesting remarks regarding the presented example are as follows:

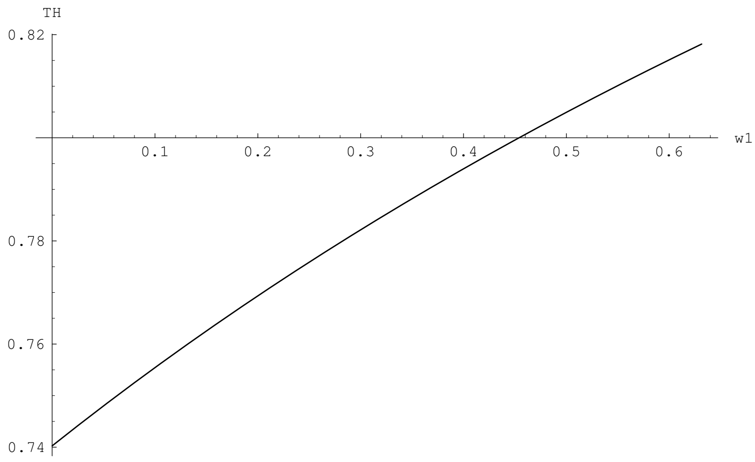

Figure 5: Example: Graphing the system (total) throughput as a function of the independent variable ω_1 and for a product-mix requirement $TH_1/TH_2 = 0.9$

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1. From a mathematical standpoint, the constraint of Equation 29, introduces some coupling between the two initially independent control variables ω_1 and ω_2, which eventually reduces the degrees of freedom of the original feasible region from 2 to 1. In other words, the feasible region for the modified problem formulation, $\mathcal{F}'$, reduces to a single curve in the original feasible region, $\mathcal{F} \equiv [0,1]^2$, which subsequently leads to the optimization of the considered formulation in an interior point of $\mathcal{F}$.

2. From a more practical standpoint, the resulting randomization of the deadlock resolution strategy is the mechanism used by the optimal policy in order to establish the required product-mix – i.e., the requirement of Equation 29 – in the system operations. In fact, it is possible that the problem formulation resulting by the addition of Constraint 25 to the original formulation of Equations 5 – 8, is infeasible. For an example, the reader can consider Equation 29 for $\xi = 2$. In this case, the accommodation of the product-mix constraint will require the control of additional transitions in the system STD, according to the randomized scheme presented above. In fact, the reader should convince herself that, in the most general case, the association of a control probability with every single

transition of the underlying STD essentially provides a *scheduling* mechanism for the considered RAS. A systematic treatment of this idea in the formal modeling framework of generalized stochastic Petri nets, and for the RAS sub-class modeling re-entrant lines, can be found in [2].

3. Finally, it should be noticed that the optimal policy for the above example was obtained at one of the two points constituting the *boundary* $\mathcal{B}$ of the feasible region $\mathcal{F}'$. In the considered problem context,[11] $\mathcal{B}$ is defined by the fact that some control variable(s) ω_{ij} takes an extreme value of its feasible interval $[0, 1]$, and further advancement beyond $\mathcal{B}$, in a way that satisfies Constraint 25, will eventually violate Constraint 8 in the original problem formulation. It turns out that the optimization of the modified problem formulation of Equations 5 – 8 and 25 on the boundary $\mathcal{B}$ of its feasible region $\mathcal{F}'$ is a more general result, as it is proven next.

Theorem 2 *Suppose that the optimization problem defined by Equations 5 – 8 and 25 has a non-empty feasible region $\mathcal{F}' \subseteq \mathcal{F} \equiv [0, 1]^{dim(\omega)}$. Then, it has an optimal solution located on the boundary $\mathcal{B}$ of $\mathcal{F}'$.*

Proof: Theorem 2 is trivially satisfied in the case that $\mathcal{B} = \mathcal{F}'$. Hence, in the following, suppose that the considered problem formulation has $\mathcal{B} \subset \mathcal{F}' \subseteq \mathcal{F} = [0, 1]^{dim(\omega)}$, and that $\omega^* \in \mathcal{F}' \backslash \mathcal{B}$ is an optimal solution for it. Then, it follows that there exists a sufficiently small $\delta > 0$, s.t. the entire neighborhood $N(\omega^*, \delta) \subseteq \mathcal{F}' \subseteq \mathcal{F}$. Moreover, the results of Section 3 regarding the monotonicity of the function $TH(\omega)$ w.r.t. the various control variables ω_{ij}, imply that there exists a *non-deteriorating* direction from ω^* to another point ω_1, located on the surface of the sphere $N(\omega^*, \delta)$. In case that the problem has a unique optimal solution, the above argument implies that $TH(\omega_1) > TH(\omega^*)$, which contradicts the original assumption that ω^* is an *optimal interior* point, and establishes the validity of the theorem.

To address the case of the existence of many optimal solutions, notice that the structure of the throughput function implied by Equation 15, combined with the existence of an optimal interior point of $\mathcal{F}'$, imply that $TH(\omega)$ is constant over $\mathcal{F}'$, and therefore, the entire set $\mathcal{B}$ constitutes a (sub-)set of optimal solutions. □

A critical reading of the proof of Theorem 2 reveals that: (i) it is an immediate consequence of the structure of the $TH(\omega; \bar{q}_{ij})$ function, established by Proposition 1, and the implied monotonicity properties; (ii) it would apply to any other set of constraints appended to the

[11] and as it can also be seen in the above example. . .

original formulation of Equations 5 – 8 (i.e., other than Constraint 25). Its availability provides an analytical insight that can be useful in the design of solution algorithms for any such extended formulations, by taking advantage of the particular form / structure of the appended constraint sets, and / or the establishment of further analytical / qualitative results for the resulting optimization problem(s). For a more concrete example on how the result of Theorem 2 can facilitate the establishment of special structure for the optimal solution space of such extended formulations, the reader is referred to [2] (c.f., proof of Theorem 1 therein).

5 Conclusions

This work has revisited the problem of the optimality of randomized deadlock avoidance policies, under the performance objective of maximizing of the system throughput, originally raised in [4, 5]. The main finding is that when the only consideration is the maximization of the overall system throughput, then the randomization of the deadlock resolution policy is not essential; i.e., there will always exist an optimal solution in which each of the problematic transitions from the safe to the unsafe operational region will always remain enabled or disabled. It was also shown, however, that randomization of the control of (some of) these transitions can provide an effective mechanism for accommodating additional operational constraints that tend to couple the underlying control variables, and thus, further constrain the problem solution space; constraints relating to the resulting product-mix were used to exemplify this effect. Finally, the presented work characterized explicitly the functional dependence of the considered objective function(s) to the underlying control variables, and in that sense, it provides broader insights concerning the control of Continuous-Time Mark Chains under steady-state operation.

The implications of this functional characterization for the broader scheduling problem of the steady-state operation of flexibly automated production systems is investigated in another part of our current work, with some initial results reported in [2]. An additional issue to be addressed in our future research concerns the management of the computational complexity of the considered problems and their optimal solution. This complexity arises from the fact that the presented Mathematical Programming (MP) formulations involve the complete enumeration of the underlying system state space, which is known to explode extremely fast for the considered class of systems (c.f., [6]). Hence, in order for the results presented in this work, as well as their potential extensions to the broader scheduling problem, to be fully exploited, we need to develop alternative more concise problem characterizations and/or computationally efficient solution techniques that can lead fast to (near-)optimal solutions for the considered performance optimization problems. The results presented herein establish a formal framework and a rigorous and insightful basis for such further computational developments.

Acknowledgments

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