Efficient Generation of Performance Bounds for a Class of Traffic Scheduling Problems*

Greyson Daugherty
Revenue Analytics

Spyros Reveliotis†
School of Industrial & Systems Engineering
Georgia Institute of Technology

Greg Mohler
Georgia Tech Research Institute
Georgia Institute of Technology

Abstract

This work seeks to develop (lower) performance bounds for a traffic scheduling problem that arises in many application contexts, ranging from industrial material handling and robotics to computer game animations and quantum computing. In a first approach, the sought bounds are obtained by applying the Lagrangian relaxation method to a MIP formulation of the considered scheduling problem that is based on a natural notion of "state" for the underlying traffic system and an analytical characterization of all the possible trajectories of this state over a predefined time horizon. But it is also shown that the corresponding "dual" problem that provides these bounds, can be transformed to a linear program (LP) with numbers of variables and constraints polynomially related to the size of the underlying traffic system and the employed time horizon in the MIP formulation. Furthermore, the derived LP formulation constitutes the LP relaxation of a second MIP formulation for the considered scheduling problem that can be obtained through an existing connection between this problem and the "integral multi-commodity flow" (IMCF) model of network optimization theory. Finally, the theoretical developments of the paper are complemented with a computational part that demonstrates the efficacy of the pursued methods in terms of the quality of the derived bounds, and their computational tractability.

Keywords: Guidepath-based Traffic Systems; Traffic Scheduling; Lagrangian Duality; Combinatorial Optimization

*This work was partially supported by NSF grant ECCS-1707695.
†Corresponding author – email: `spyros@isye.gatech.edu`, phone: 404-894-6608

1 Introduction

The basic traffic scheduling problem underlying the developments that are presented in this paper, can be briefly described as follows: Given a set of "agents", $\mathcal{A}$, that circulate on the edges and/or the vertices of a connected graph $G = (V, E)$, to be called the "(supporting) guidepath network", we want to advance these agents from their current locations to some destination locations in the minimum possible time, while observing a number of regulations regarding the edge and/or the vertex occupancy by the traveling agents.[1]

From a practical standpoint, the above problem is motivated by the need to study the dynamics of traffic that is evolving in highly constricted environments. More specifically, particular instantiations of this scheduling problem have been investigated by: (i) the artificial intelligence (AI) and the robotics communities under the name of the multi-agent path planning (MAPP) problem (e.g.,[31, 29, 39, 20]); (ii) the industrial engineering and the operations research communities in the operational context of various unit-load material handling systems (MHS) (e.g., [13, 18, 15, 10, 23, 9]); (iii) the computer scientists and the theoretical physicists that deal with the motion of the ionized atoms (or "qubits") that are the elementary information carriers in the context of quantum computing [22, 5]; and (iv) the computer game industry in its effort to support complex animated computer graphics [33].

The restrictions that are imposed by the aforementioned operational environments on the traveling agents that circulate in them, define various notions of "conflict" for these agents, and, in certain cases, they can even impair the agents' ability to reach their intended destinations. As a result, for many of the considered scheduling formulations, even the more basic tasks of (a) assessing their feasibility, and (b) constructing just a feasible traffic schedule, can be very challenging [36, 16, 2, 28, 27, 40]. Furthermore, the optimization problem that is defined by these scheduling formulations, will typically fall into the class of NP-hard problems [38, 20]. Hence, at the end, these scheduling problems are usually addressed through heuristics and approximating methods that aim for near-optimal solutions. Finally, an additional complicating feature of the aforementioned traffic scheduling problems is that the posed transport requirements are emerging in a very dynamic manner, that necessitates the iterative re-computation of an optimized traffic schedule.

When viewed from a more technical standpoint, most of the current literature on the aforementioned traffic scheduling problems can be perceived as an endeavor to adapt classical ideas and techniques coming from combinatorial optimization [24] and scheduling theory [21, 25] to

[1] The time that it is required by any feasible traffic schedule for the considered problem to move all the traveling agents from their current locations to their destinations, is known as the "makespan" of this schedule; hence, in more technical terms, our problem is a "min makespan" scheduling problem. We shall provide a detailed positioning of this problem in later parts of this work.

this particular problem setting. Hence, a large part of the proposed heuristics and algorithms constitute "local-search"-based schemes [24] and variations of the popular A^* algorithm [21] adapted to the considered problem context. Many of these schemes are also augmented with further decomposing or filtering steps that seek to enhance the computational efficiency of the resulting algorithms, and also the feasibility and the efficiency of the derived solutions. Some indicative samples of these lines of work can be found in [31, 32, 35, 29, 34, 30, 9]. Also, the recent publication of [9] provides a very systematic and quite comprehensive survey of the existing literature on the effective and efficient traffic management of the various classes of the guidepath-based traffic systems considered in this work.

Of particular interest and affinity to this work, are some recent developments that have appeared in [39, 20], and connect the considered traffic scheduling problems to the notion of the "integral multi-commodity flow (IMCF)" and the corresponding network-flow optimization theory [1]. The realization of this connection has enabled the effective resolution of the particular traffic scheduling problems that are addressed in [39, 20] through integer programming (IP) formulations [37]. More specifically, these IP formulations either provide a complete representation of the addressed traffic scheduling problem, or they are embedded as components in more general solution algorithms for these problems; and especially in the latter case, they have resulted in the expedient computation of near-optimal solutions for many instances from the considered scheduling problems that involve a large number of agents and a complex structure for the underlying guidepath network.

This paper seeks to complement the aforementioned developments by taking a closer look at the combinatorial structure of the considered traffic scheduling problems, and focusing primarily on the ability of this structure to provide good quality (lower) bounds for their optimal objective value. The importance of the availability of good-quality bounds for hard scheduling (or more general combinatorial optimization) problems is well understood in the corresponding optimization theory; these bounds enable the assessment of the extent of the sub-optimality of the solutions that are derived through the heuristic methods that were mentioned in the previous paragraphs, and they can also guide and streamline the search for good solutions by steering the search process away from regions of the underlying solution space that are of poor quality [11, 3]. In certain, although rarer, cases, the computation of these bounds can also reveal additional structural information for the synthesis of efficient solutions for the considered optimization problem [3, 14].

From a methodological standpoint, the starting point for the derivation of the bounds that are sought in this paper, is a complete mathematical programming (MP) formulation of the considered optimization problem. As already mentioned, the traffic scheduling problems considered in this work can be formulated as a Mixed Integer Program (MIP) [37]. Once such a

MIP formulation has been derived, there are two possible approaches to derive the sought lower bounds: (a) One first bound can be obtained by relaxing the integrality property of the integer (actually, binary) variables that model the more combinatorial elements of the considered problem; the resulting formulation is known as the "Linear Programming (LP) relaxation" of the original MIP, and the corresponding bounds are referred to as the LP bounds. (b) The second line of computation of the sought bounds is by "softening" some of the harder constraints in the original MIP formulation, bringing these constraints into the original objective function as additional terms that will "penalize" the violation of these constraints by any contemplated solution; the formulation that results from this modification is known as the "Lagrangian relaxation" of the original MIP formulation, and the theory that deals with the systematic derivation of the corresponding bounds and the analysis of their quality, is known as "(Lagrangian) duality" theory [11, 3].

Furthermore, two additional important facts that have been established by Lagrangian duality theory and are at the center of this work, are the following: (i) For MIP formulations of the type considered in this work, Lagrangian duality can provide tighter bounds than the corresponding LP bounds, in general [12]. (ii) The computation of the tightest Lagrangian-duality bounds that result from a given selection of the relaxed constraints, boils down to the solution of a concave optimization problem that is known as the corresponding "dual" problem. But this concave optimization problem involves a non-smooth objective function and requires the employment of sub-gradient optimization techniques; hence, its solution can be a slow and unstable process, especially for large-scale instantiations of the considered problems [3]. These two facts subsequently imply that the quality – or, the "tightness" – of the derived bounds and the computational complexity that is involved in their derivation, can be contingent upon (a) the starting MIP formulation, and (b) the employed method for the derivation of these bounds.

In view of the above remarks, the main contributions of this paper can be stated as follows:

1. First we provide a MIP formulation for the traffic scheduling problems that are considered in this work that constitutes a "canonical" such formulation for these problems. More specifically, the primary decision variables that are employed by this MIP formulation are suggested by a natural notion of *"state"* for the considered traffic systems, and, together with the constraints of this formulation, they define a straightforward representation of the evolution – or of the "trajectory" – of this state under any contemplated feasible schedule. Hence, the notions and the perspectives that are employed by this formulation parallel the notions and ideas that underlie the formulations of more general optimal control problems.

2. The aforementioned MIP formulation is subsequently used in order to derive tight lower

bounds for the considered scheduling problem through the Lagrangian duality theory that was discussed in the previous paragraphs. More specifically, these bounds are generated by relaxing some hard constraints in the original MIP formulation that "couple" the routing and scheduling decisions among the traveling agents. But in an additional important step, it is shown that the corresponding "dual" problem that will compute the tightest possible value among this set of bounds, possesses special structure that allows the eventual reformulation of this problem as an LP. Furthermore, the numbers of variables and constraints of this LP are polynomially related to the system parameters $|\mathcal{A}|$, $|E|$, and the time horizon T that is employed in the presented formulation of the underlying traffic scheduling problem. Hence, the hard non-smooth concave optimization problem that was mentioned in the previous paragraphs, can be solved in our case through canned LP software in an efficient and very robust manner.

3. We also provide an explanation of the linearizing result that was mentioned in item #2 above, by interpreting the obtained LP formulation as the LP relaxation of another MIP formulation of the considered scheduling problem. In fact, it turns out that this new MIP formulation of the considered scheduling problem is naturally motivated and defined by the IMCF structure that is present in the dynamics of the considered traffic systems. Hence, besides their explanatory role along the aforementioned lines, the results of this part of our work also connect the developments of this paper to the recent developments and insights of [39, 20].[2] They also establish that the bounds resulting from the alternative modeling approach of the considered traffic scheduling problems that recognizes the IMCF structure present in them, are of exactly the same quality and informational content with the bounds obtained through the MIP formulation of item #1.

4. Finally, a last part of the paper presents the results of a set of computational experiments that (i) demonstrate the computational potency and efficiency in the derivation of the sought bounds that is provided by the paper developments, and, even more importantly, (ii) help us assess the quality of the bounds that can be obtained from these developments, in terms of their proximity to the optimal objective value of the underlying MIP formulation.

With this positioning of the paper content and its intended contribution, the rest of it is organized as follows: Section 2 provides a systematic description of the considered traffic scheduling problems and their first MIP formulation that was described in item #1. Section 3 presents the Lagrangian relaxation that is proposed for the MIP formulation of Section 2, formulates

[2] We emphasize, however, that our results have been developed in the Ph.D. thesis program of the first author independently from, and in parallel to, the corresponding developments of [39, 20].

the corresponding "dual" problem, and establishes some important properties for this problem. Section 4 derives a series of LP (re-)formulations of the "dual" problem of Section 3, and establishes formally the equivalence of these LP formulations to the original "dual" problem. Section 5 introduces the alternative MIP formulation for the considered traffic scheduling problem, and provides a more intuitive explanation for the results of Section 4 by establishing the equivalence of the LP relaxation of this new MIP formulation to the LP formulation of the Lagrangian "dual" problem of Section 3. Section 6 presents the numerical results regarding the computational tractability of the proposed LP-based approach for the solution of the "dual" problem, and the tightness of the derived bounds. Finally, Section 7 concludes the paper and discusses some directions for future work. We also notice, for completeness, that a preliminary, more concise version of some of the presented results have appeared in [6, 8].

2 The traffic scheduling problem considered in this work: Detailed problem description and its first MIP formulation

In this section we provide a formal characterization of the traffic systems and the corresponding traffic scheduling problem that are the focus of this work. Some of the concepts and the terminology introduced in the subsequent discussion are motivated from our experience with these scheduling problems in the context of the MHS and the quantum-computing operations that were mentioned in the introductory section. But the overall positioning of the problem, and the corresponding modeling assumptions, have been kept at a level of generality that renders the presented developments easily transferrable to the other variations of these traffic scheduling problems that were cited in the introductory section.

An abstracting definition of the guidepath-based traffic systems considered in this work: The traffic system that is considered in this work can be formally abstracted as follows: The system consists of a guidepath graph $G = (V, E)$ that is traversed by a set of agents, $\mathcal{A}$. Graph G is assumed to be connected and undirected. The edges $e \in E$ of G model the "zoning" of the underlying quidepath network, i.e., the segmentation of this network into a set of locations that are allocated sequentially and exclusively to their occupying agents in order to perform their traveling through this network. Hence, each edge $e \in E$ can be traversed by a traveling agent $a \in \mathcal{A}$ in either direction, but they can hold no more than one agent at any time.

Each agent $a \in \mathcal{A}$ initially occupies an edge s_a of G and also has associated with it a destination edge d_a. We want to define a set of routes that will take each agent a from its current location s_a to its destination location d_a in a way that (i) observes the exclusive occupancy of the system zones by the traveling agents, and (ii) optimizes an objective function that is defined with respect to (w.r.t.) certain attributes of these routes. Different variations of the above problem can be obtained through the detailing of: (a) the dynamics that define the agent motion within a zone or their transitions between two neighboring zones; (b) the protocol that coordinates the zone allocation to the traveling agents; and (c) the particular objective to be pursued by the selected routes.

For the expository needs of this work, we define the additional problem elements that were listed in the previous paragraph through the following assumptions: In our problem formulation, we assume that the edge traversal time is uniform for all agent-edge pairs $(a, e) \in \mathcal{A} \times E$, and this time constitutes a natural "time unit" that discretizes the dynamics of the underlying traffic.[3] In the resulting discretized traffic model, it is further assumed that an agent a can transition from an edge e, that it occupies at some period t, to a neighboring edge e', to be occupied at period $t + 1$, only if edge e' is unoccupied at period t.[4] Also, while an edge $e \in E$ can be traversed by an agent $a \in \mathcal{A}$ in any direction, in the rest of this work we also assume that, once agent a has entered edge e, it cannot reverse the direction of its motion on this edge. To capture this sense of direction of the agent motion in its current edge, in our analytical formulations of the considered scheduling problem, we shall represent each undirected edge $e \in E$ in the original guidepath network, that connects some vertices v_i and v_j, with the pair of directed edges (v_i, v_j) and (v_j, v_i); this replacement will turn the original graph G into a directed graph. Furthermore, in this extended representation, for any edge $e = (v_i, v_j)$, we shall also use the notation of $\bar{e}$ to denote the "reverse" edge (v_j, v_i). Finally, in our formulation of the considered scheduling problem, we shall seek to minimize the "makespan" of the employed schedule, i.e., the time by which every agent $a \in \mathcal{A}$ has reached its destination d_a.[5]

Finally, we also notice, for completeness, that in many practical application contexts, the traffic scheduling problem that was outlined in the previous paragraphs will constitute the "core" sub-problem that must be repetitively formulated and solved in the context of an MPC / "rolling-horizon" scheme [17] that will address more complicated routing schemes and/or more dynamically evolving transport requirements for the considered traffic systems. The complete definition of such an MPC scheme able to ensure the efficiency and the liveness of the generated traffic, depends on the aforementioned operational assumptions that define the

[3]While facilitating the presentation of the subsequent developments, the presumed uniformity of the zone traversal times is not restrictive, since one can adapt the presented developments to the non-uniform case by using the greatest common divisor of the various zone traversal times as the discretizing time unit.

[4]As acknowledged in [39], this is a typical requirement for most models of guidepath-based traffic systems. In the context of zone-controlled until-load MHS [13] and of the qubit transport systems that are employed by quantum computing [5], this requirement is motivated by the need to ensure collision-freedom for the traveling agents during the unobservable, transitional phase between time periods t and $t + 1$.

[5]Besides its practical relevance in many applications, the selection of the schedule makespan as the employed objective function also expresses our intention to address one of the most difficult variations of the considered scheduling problem, since this criterion is a nonlinear function of the dynamics of the underlying traffic.

agent maneuverability and the employed zone allocation protocol; a more expansive discussion on this MPC scheme and a systematic treatment of the aforementioned dependencies can be found in [9, 26].

Formulating the considered traffic scheduling problem as a Mixed Integer Program: Next, we develop the first Mixed Integer Programming (MIP) [37] formulation for the traffic scheduling problem that was detailed in the previous part of this section. The notation and the decision variables that are employed in this formulation are defined as follows:

Notation

- $V = \{v_1, v_2, ..., v_m\}$: Guidepath-graph vertices
- $E = \{e_1, e_2, ..., e_n\}$, with $e_l = (v_i, v_j)$ $\forall l \in \{1, ..., n\}$: Guidepath-graph edges (or "zones")
- $\mathbb{T}$ $(n \times n)$: A binary matrix expressing the agent transitional dynamics on the guidepath graph; $\mathbb{T}_{i,j} = 1$ iff a direct transition from e_i to e_j is allowed. We also set $\mathbb{T}_{i,i} = 1$ $\forall i$ s.t. $e_i \in E$
- $\bar{e} = (v_j, v_i)$: Complementary edge of edge $e = (v_i, v_j)$
- ${}^{\bullet}e_l = \{e_q \in E : \mathbb{T}_{q,l} = 1 \wedge q \neq l\}$: The set of input edges for e_l, $\forall e_l \in E$
- $e_l^{\bullet} = \{e_q \in E : \mathbb{T}_{l,q} = 1 \wedge q \neq l\}$: The set of output edges for e_l, $\forall e_l \in E$
- $\mathcal{A} = \{a_1, a_2, ..., a_K\}$: The set of traveling agents
- d_a: Destination edge for agent $a \in \mathcal{A}$
- s_a: Starting edge for agent $a \in \mathcal{A}$; this edge also specifies the initial orientation for the agent motion
- T: An upper bound on the required transport time, across all agents (and therefore an upper bound on the optimal value of the objective function).
- $t \in \mathcal{T} = \{0, 1, ..., T\}$: Time index

Decision Variables

- $\forall a \in \mathcal{A}$, $\forall e \in E$, $\forall t \in \mathcal{T}$, $x_{a,e,t} \in \{0, 1\}$ indicates whether, in the derived traffic schedule, agent a is located on the directed edge e at timestep t; these decision variables define a natural notion of *"state"* for the underlying traffic over the considered time horizon

T, and they constitute the primary decision variables of the considered traffic scheduling problem. Also, for notational convenience, in the following, we shall denote by $\boldsymbol{x}$ the vector that collects all the variables $x_{a,e,t}$.

- w: An auxiliary variable that will represent the "makespan" – i.e., the total time to completion – of the optimal traffic schedule.

The MIP formulation itself is as follows:

$$\min w \tag{1}$$

s.t.

$$\forall a \in \mathcal{A},\ \forall t \in \mathcal{T},\quad \sum_{e \in E} x_{a,e,t} = 1 \tag{2}$$

$$\forall a \in \mathcal{A},\ \forall e \in E,\quad x_{a,e,0} = I_{\{e=s_a\}} \tag{3}$$

$$\forall a \in \mathcal{A},\quad x_{a,d_a,T} = 1 \tag{4}$$

$$\forall a \in \mathcal{A},\ \forall t \in \mathcal{T} \setminus \{T\},\quad x_{a,d_a,t+1} \geq x_{a,d_a,t} \tag{5}$$

$$\forall a \in \mathcal{A},\ \forall e \in E,\ \forall t \in \mathcal{T} \setminus \{T\}, x_{a,e,t} \leq x_{a,e,t+1} + \sum_{e' \in e^{\bullet}} x_{a,e',t+1} \tag{6}$$

$$\forall e = (v_i, v_j) \in E \text{ s.t. } i < j,\ \forall t \in \mathcal{T} \setminus \{0, T\}, \sum_{a \in \mathcal{A}} (x_{a,e,t} + x_{a,\bar{e},t}) \leq 1 \tag{7}$$

$$\forall a \in \mathcal{A},\ \forall e \in E,\ \forall t \in \mathcal{T} \setminus \{0\}, x_{a,e,t} + \sum_{a' \in \mathcal{A}: a' \neq a} (x_{a',e,t-1} + x_{a',\bar{e},t-1}) \leq 1 \tag{8}$$

$$\forall a \in \mathcal{A},\ w \ \geq \ \sum_{t=0}^{T} (1 - x_{a,d_a,t}) \ = \ T + 1 - \sum_{t=0}^{T} x_{a,d_a,t} \tag{9}$$

$$\forall a \in \mathcal{A},\ \forall e \in E,\ \forall t \in \mathcal{T},\quad x_{a,e,t} \in \{0, 1\} \tag{10}$$

A brief explanation of the constraints appearing in the above formulation is as follows: Constraint (2) imposes the requirement that all agents must occupy one and only one position at any time period. On the other hand, Constraint (3) places the agents $a \in \mathcal{A}$ at their initial zones s_a at time $t = 0$; in particular, the notation $I_{\{e=s_a\}}$ that appears in the right-hand-side of this constraint denotes an indicator variable that is equal to 1 if the condition $e = s_a$ is true. Constraint (4) requires that every agent must reach its destination edge, d_a, within the provided time horizon, while Constraint (5) further stipulates that agents cannot leave their destination edges after reaching them.[6] Constraint (6) enforces the fact that agents can only transition to adjacent, directionally compatible edges, and Constraint (7) prevents the concurrent occupancy of an edge by more than one agent.[7] Constraint (8) enforces the additional requirement that an agent can enter an edge at period t only if this edge was empty during the previous period, $t-1$. As discussed in the previous part of this section, this requirement is imposed due to safety considerations, and it further implies that agents located on neighboring edges can neither "swap" their edges nor "dovetail" each other in their transitions among a cascade of such edges. Constraint (9) together with Equation (1) define the objective of the considered formulation as the minimization of the makespan of the traffic schedule to be generated for the traveling agents. Finally, Constraint (10) specifies the binary nature of all decision variables $x_{a,e,t}$. We also notice that the remaining decision variable w is a free variable.

The MIP formulation of Eqs (1)–(10) provides a succinct characterization of the considered scheduling problem, but it becomes computationally intractable even for moderately sized instantiations of this problem. These computational challenges are further aggravated by the "on-line" / "real-time" nature of the considered problem, and the strict time budgets that this feature implies for the involved computations. Hence, in practice, the considered scheduling problem typically will be solved by some heuristic algorithm. In the rest of this work, we use the MIP formulation of Eqs (1)–(10) primarily as a starting point in order to compute high-quality lower bounds for its optimal objective value (i.e., for the optimal makespan of the corresponding scheduling problem).[8]

3 A Lagrangian relaxation for the MIP of Section 2, and the corresponding "dual" problem

The proposed Lagrange relaxation and the corresponding "dual" problem: It should be evident to the astute reader that the increased complexity of the MIP formulation of Eqs (1)–(10) arises

[6]This constraint is dictated by the broader logic that defines the MPC scheme that provides context for the considered MIP formulation.

[7]In the statement of this constraint, we further assume that the constraint is observed by the problem data that specify the initial positions, s_a, and the destinations, d_a, of the agents $a \in \mathcal{A}$.

[8]We also notice, for completeness, that the MIP formulation of Eqs (1)–(10) can be infeasible for some of its instantiations, and the assessment of the corresponding (in-)feasibility is a hard problem in itself. In general, the feasibility of the considered MIP will depend on (i) the topology of the underlying guidepath network and the relative positioning of the edges s_a and d_a, $a \in \mathcal{A}$, in this topology, as well as (ii) the selection of the parameter T. Determining feasibility w.r.t. the first of the above two elements is a hard "reachability" problem that should be addressed in the context of the "untimed" dynamics of the underlying traffic system, using, for instance, some "linguistic" modeling framework for these dynamics, like automata theory or Petri nets [4]. On the other hand, for feasible problem instances w.r.t. criterion (i), a pertinent T value that will not compromise this feasibility, can be obtained through the solution of the corresponding scheduling problem by a heuristic method.

from the restrictions that are imposed by Constraints (7)–(9), since, otherwise, the traveling agents could be routed to their target destinations, $d_a,\ a \in \mathcal{A}$, through any shortest path in the underlying guidepath network that connects the agent current location, s_a, to the corresponding destination, d_a. Hence, a pertinent Lagrangian relaxation for this MIP will relax the "coupling" Constraints (7)–(9). For a detailed specification of this relaxation, let us denote the three vectors of the Lagrange multipliers that correspond to each of these three constraint sets, as follows:

- $\boldsymbol{\lambda}$: vector of Lagrange multipliers for constraint set (7);
- $\boldsymbol{\mu}$: vector of Lagrange multipliers for constraint set (8);
- $\boldsymbol{\nu}$: vector of Lagrange multipliers for constraint set (9).

In the following, we shall refer to the specific elements of the constraint sets (7)–(9), and the corresponding Lagrange multipliers, by the particular index tuples that define these constraints within the corresponding constraint set; hence, for instance, the elements of vector $\boldsymbol{\lambda}$ will be represented by $\lambda_{e,t},\ e \in \{(v_i, v_j) \in E : i < j\},\ t \in \mathcal{T} \setminus \{0, T\}$. Under this notational convention, the defined Lagrangian function can be written as follows:

$$\begin{aligned} L(\boldsymbol{x}, w; \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \ \equiv\ & w + \sum_{\{e \in E: v_i < v_j\}} \sum_{t \in \mathcal{T} \setminus \{0,T\}} \lambda_{e,t} \left[\sum_{a \in \mathcal{A}} (x_{a,e,t} + x_{a,\bar{e},t}) - 1 \right] + \\ & + \sum_{a \in \mathcal{A}} \sum_{e \in E} \sum_{t \in \mathcal{T} \setminus \{0\}} \mu_{a,e,t} \left[x_{a,e,t} + \sum_{\{a' \in \mathcal{A}: a' \neq a\}} (x_{a',e,t-1} + x_{a',\bar{e},t-1}) - 1 \right] + \\ & + \sum_{a \in \mathcal{A}} \nu_a \left[T + 1 - w - \sum_{t \in \mathcal{T}} x_{a,d_a,t} \right] \end{aligned} \quad (11)$$

with

$$\boldsymbol{\lambda} \geq \mathbf{0}; \quad \boldsymbol{\mu} \geq \mathbf{0}; \quad \boldsymbol{\nu} \geq \mathbf{0} \quad (12)$$

Furthermore, for any arbitrary pricing of the Lagrange multipliers $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, the "relaxed" version of the MIP formulation of Section 2 can be expressed as follows:

$$\begin{aligned} \theta(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \ \equiv\ \min_{\boldsymbol{x}, w} \Bigg\{ & w + \sum_{\{e \in E: v_i < v_j\}} \sum_{t \in \mathcal{T} \setminus \{0,T\}} \lambda_{e,t} \left[\sum_{a \in \mathcal{A}} (x_{a,e,t} + x_{a,\bar{e},t}) - 1 \right] + \\ & + \sum_{a \in \mathcal{A}} \sum_{e \in E} \sum_{t \in \mathcal{T} \setminus \{0\}} \mu_{a,e,t} \left[x_{a,e,t} + \sum_{\{a' \in \mathcal{A}: a' \neq a\}} (x_{a',e,t-1} + x_{a',\bar{e},t-1}) - 1 \right] + \\ & + \sum_{a \in \mathcal{A}} \nu_a \left[T + 1 - w - \sum_{t \in \mathcal{T}} x_{a,d_a,t} \right] \Bigg\} \end{aligned} \quad (13)$$

s.t. the primal constraint sets (2)–(6) and (10)

Function $\theta(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is known as the *"dual" function* for the original MIP formulation of Section 2 (under the particular selection of the relaxed constraints) [3]. Furthermore, since the minimization problem in the right-hand-side of Eq. (13), that defines this function, is a relaxation of the original MIP formulation, it follows that function $\theta(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$ provides a lower bound to the optimal value for the MIP formulation of Section 2, for any values of its variables $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. Naturally, we are interested in obtaining the tightest possible lower bound that can be provided by $\theta(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$. This bound can be obtained by solving the following optimization problem:

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}} \theta(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \text{ s.t. Eq. (12)} \tag{14}$$

The above formulation is known as the *"dual" problem* for the original MIP formulation of Section 2 [3].

A structural analysis of the considered "dual" problem: Next, we proceed to reveal additional structure in the Lagrangian function of Eq. (11) that will prove particularly useful for the efficient evaluation of the dual function $\theta(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$, for any given vector $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu})$, and for the solution of the corresponding dual problem. By rearranging its terms, the Lagrangian function of Eq. (11) can be rewritten as follows:

$$\begin{aligned} L(\boldsymbol{x}, w; \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) \; = \; & \sum_{a\in\mathcal{A}} \nu_a(T+1) - \Bigg[\Bigg(\sum_{\{e\in E: v_i<v_j\}} \sum_{t\in\mathcal{T}\setminus\{0,T\}} \lambda_{e,t}\Bigg) + \Bigg(\sum_{a\in\mathcal{A}}\sum_{e\in E}\sum_{t\in\mathcal{T}\setminus\{0\}} \mu_{a,e,t}\Bigg)\Bigg] \\ & + w(1-\sum_{a\in\mathcal{A}}\nu_a) \; + \; \sum_{a\in\mathcal{A}}\Bigg\{\sum_{t\in\mathcal{T}\setminus\{0,T\}}\Bigg(\sum_{\{e\in E: v_i<v_j\}}\Bigg[\lambda_{e,t}+\mu_{a,e,t}+ \\ & \sum_{a'\in\mathcal{A}:a'\neq a}(\mu_{a',e,t+1}+\mu_{a',\bar{e},t+1})\Bigg]x_{a,e,t} + \sum_{\{e\in E: v_i>v_j\}}\Bigg[\lambda_{\bar{e},t}+\mu_{a,e,t}+\sum_{a'\in\mathcal{A}:a'\neq a}(\mu_{a',e,t+1}+ \\ & \mu_{a',\bar{e},t+1})\Bigg]x_{a,e,t}\Bigg) + \sum_{e\in E}\Bigg[\sum_{a'\in\mathcal{A}:a'\neq a}(\mu_{a',e,1}+\mu_{a',\bar{e},1})\Bigg]x_{a,e,0} + \sum_{e\in E}\mu_{a,e,T}x_{a,e,T} \; - \; \nu_a\sum_{t\in\mathcal{T}}x_{a,d_a,t}\Bigg\} \end{aligned} \tag{15}$$

Then, setting

$$\Delta_{\boldsymbol{\lambda},\boldsymbol{\mu}} \; \equiv \; -\Bigg[\Bigg(\sum_{\{e\in E: v_i<v_j\}} \sum_{t\in\mathcal{T}\setminus\{0,T\}} \lambda_{e,t}\Bigg) + \Bigg(\sum_{a\in\mathcal{A}}\sum_{e\in E}\sum_{t\in\mathcal{T}\setminus\{0\}} \mu_{a,e,t}\Bigg)\Bigg]; \tag{16}$$

$$C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,t} \equiv \begin{cases} \lambda_{e,t} + \mu_{a,e,t} + \sum_{a'\in\mathcal{A}:a'\neq a}(\mu_{a',e,t+1} + \mu_{a',\bar{e},t+1}), \\ \forall a \in \mathcal{A},\ \forall t \in \mathcal{T}\setminus\{0,T\},\ \forall e \in E : v_i < v_j; \\ \lambda_{\bar{e},t} + \mu_{a,e,t} + \sum_{a'\in\mathcal{A}:a'\neq a}(\mu_{a',e,t+1} + \mu_{a',\bar{e},t+1}), \\ \forall a \in \mathcal{A},\ \forall t \in \mathcal{T}\setminus\{0,T\},\ \forall e \in E : v_i > v_j; \end{cases} \quad (17)$$

$$C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,0} \equiv \sum_{a'\in\mathcal{A}:a'\neq a}(\mu_{a',e,1} + \mu_{a',\bar{e},1}),\ \forall a \in \mathcal{A},\ \forall e \in E; \quad (18)$$

$$C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,T} \equiv \mu_{a,e,T},\ \forall a \in \mathcal{A},\ \forall e \in E; \quad (19)$$

we obtain the following expression for the Lagrangian function:

$$\begin{aligned} L(\boldsymbol{x}, w; \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) &= \sum_{a\in\mathcal{A}} \nu_a(T+1) + \Delta_{\boldsymbol{\lambda},\boldsymbol{\mu}} + w(1 - \sum_{a\in\mathcal{A}} \nu_a) \\ &+ \sum_{a\in\mathcal{A}} \left\{ \sum_{e\in E} \sum_{t\in\mathcal{T}\setminus\{0,T\}} C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,t} x_{a,e,t} + \sum_{e\in E} C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,0} x_{a,e,0} + \sum_{e\in E} C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,T} x_{a,e,T} - \sum_{t\in\mathcal{T}} \nu_a x_{a,d_a,t} \right\} = \\ &\sum_{a\in\mathcal{A}} \nu_a(T+1) + \Delta_{\boldsymbol{\lambda},\boldsymbol{\mu}} + w(1 - \sum_{a\in\mathcal{A}} \nu_a) + \sum_{a\in\mathcal{A}} \left[\sum_{e\in E} \sum_{t\in\mathcal{T}} C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,t} x_{a,e,t} - \nu_a \sum_{t\in\mathcal{T}} x_{a,d_a,t} \right] \end{aligned} \quad (20)$$

Also, from Eq. (20), we get the following representation of the dual function:

$$\begin{aligned} \theta(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) &= (T+1) \sum_{a\in\mathcal{A}} \nu_a + \Delta_{\boldsymbol{\lambda},\boldsymbol{\mu}} + \min_{\boldsymbol{x},w} \left\{ w(1 - \sum_{a\in\mathcal{A}} \nu_a) \right. \\ &\left. + \sum_{a\in\mathcal{A}} \left[\sum_{e\in E} \sum_{t\in\mathcal{T}} C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,t} x_{a,e,t} - \nu_a \sum_{t\in\mathcal{T}} x_{a,d_a,t} \right] \right\} \end{aligned} \quad (21)$$

s.t. the primal constraint sets (2)–(6) and (10)

The minimization problem that appears in the right-hand-side of Eq. (21), when combined with the free nature of variable w in this problem, further imply the following proposition:

Proposition 1 *At any optimal solution* $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{\nu}^*)$ *of the dual problem that is defined by Eqs (14), (12), it must hold that*

$$\sum_{a\in\mathcal{A}} \nu^*_a = 1.0 \quad (22)$$

Proof: If $\sum_{a\in\mathcal{A}} \nu^*_a < 1.0$ (resp., $\sum_{a\in\mathcal{A}} \nu^*_a > 1.0$), we can make the right-hand-side of Eq. (21)

arbitrarily small by setting the value of the free variable w arbitrarily small (resp., arbitrarily large). $\square$

To facilitate the subsequent discussion, let us define the finite set X by

$$X \;\equiv\; \Big\{\boldsymbol{x} \text{ satisfying primal constraint sets (2)–(6) and (10)}\Big\} \tag{23}$$

Then, an immediate implication of Proposition 1 is the reduction of the dual problem of Eq. (21) to the following simpler form:

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}} \theta(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}) \;\equiv\; (T+1) + \Delta_{\boldsymbol{\lambda},\boldsymbol{\mu}} + \min_{\boldsymbol{x}\in X}\left\{\sum_{a\in\mathcal{A}}\left[\sum_{e\in E}\sum_{t\in\mathcal{T}} C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,t} x_{a,e,t} - \nu_a \sum_{t\in\mathcal{T}} x_{a,d_a,t}\right]\right\} \tag{24}$$

s.t.

$$\boldsymbol{\lambda} \geq \mathbf{0};\quad \boldsymbol{\mu} \geq \mathbf{0};\quad \boldsymbol{\nu} \geq \mathbf{0};\quad \sum_{a\in\mathcal{A}} \nu_a = 1.0 \tag{25}$$

A first remark regarding the above formulation is that the minimization problem that defines the eventually optimized dual function $\theta(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})$ is *separable* with respect to the agent set $\mathcal{A}$, a fact that will have some very significant implications for the subsequent developments.

A second important implication of the formulation of Eqs (24)–(25) is that function $\theta(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})$ can also be set in the form

$$\theta(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}) = \min_{\boldsymbol{x}\in X}\Big\{(\boldsymbol{\lambda}^T,\boldsymbol{\mu}^T,\boldsymbol{\nu}^T)\cdot \boldsymbol{g}(\boldsymbol{x})\Big\} + (T+1) \tag{26}$$

where, for any given $\boldsymbol{x} \in X$, the components of the corresponding vector $\boldsymbol{g}(\boldsymbol{x})$ are linear functions of $\boldsymbol{x}$ that are completely defined by (i) the structure of the function $\theta(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu})$ defined in Eq. (24), and (ii) Eqs (16)–(19) that define the involved quantities $\Delta_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ and $C^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{a,e,t}$. Eq. (26) reveals the structure of θ as a *polyhedral concave* function, and therefore, non-differentiable at the points of intersection of its defining hyperplanes. Hence, the solution of the dual problem that is defined by Eqs (24)–(25) must rely on subgradient optimization methods. In general, the application of subgradient optimization methods to formulations that are generated through duality theory, can be challenged by slow and non-monotonic convergence. But, in the next sections, we show that the optimization problem of Eqs (24)–(25) can be re-formulated as an LP that is effectively solvable by commercial LP solvers; hence, for the considered class of scheduling problems, the corresponding dual problem can be solved exactly in finite time, in a very robust and efficient manner.

4 Reformulating the considered "dual" problem as an LP

A first linearization of the "dual" problem of Eqs (24)–(25): To derive the main results of this paper, we start with the observation that the representation of the "dual" function according to Eq. (26) allows us to express the "dual" problem of Eqs (24)–(25), modulo the term $T+1$ in the right-hand-side of Eq. (24), as the following LP:

$$\max_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu},u} \; u \tag{27}$$

s.t.

$$\forall \boldsymbol{x} \in X, \quad u \;\leq\; \boldsymbol{g}(\boldsymbol{x})^T \cdot (\boldsymbol{\lambda}^T, \boldsymbol{\mu}^T, \boldsymbol{\nu}^T)^T \tag{28}$$

$$\sum_{a \in \mathcal{A}} \nu_a = 1.0 \tag{29}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}; \quad \boldsymbol{\mu} \geq \mathbf{0}; \quad \boldsymbol{\nu} \geq \mathbf{0} \tag{30}$$

Furthermore, from standard LP duality theory [19], the optimal value of the above LP can also be obtained by solving its dual LP, that has the following form:

$$\min_{\boldsymbol{\gamma},\boldsymbol{z},\psi} \psi \tag{31}$$

s.t.

$$[-I \;\; \mathbf{1}_{\boldsymbol{\nu}}] \cdot \begin{bmatrix} \boldsymbol{z} \\ \psi \end{bmatrix} \;=\; \sum_{\boldsymbol{x} \in X} \boldsymbol{\gamma}_{\boldsymbol{x}} \cdot \boldsymbol{g}(\boldsymbol{x}) \tag{32}$$

$$\sum_{\boldsymbol{x} \in X} \boldsymbol{\gamma}_{\boldsymbol{x}} = 1.0 \tag{33}$$

$$\boldsymbol{\gamma} \geq \mathbf{0}; \quad \boldsymbol{z} \geq \mathbf{0} \tag{34}$$

In the above LP formulation, the nonnegative vector $\boldsymbol{\gamma}$ collects the dual variables that correspond to the constraints of Eq. (28) in the primal LP, and the free variable ψ is the dual variable corresponding to the constraint of Eq. (29). On the other hand, vector $\boldsymbol{z}$ is a set of "slack" variables that converts the constraint of Eq. (32) to an "equality" constraint. Furthermore, in Eq. (32), I denotes the identity matrix of dimensionality k equal to the total number of Lagrange multipliers $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, and $\mathbf{1}_{\boldsymbol{\nu}}$ is a k-dimensional binary (column) vector with its unit elements placed at the components that correspond to Lagrange multipliers $\boldsymbol{\nu}$.

While the formulations of Eqs (27)–(30) and Eqs (31)–(34) provide valid LP representations for the dual problem of Eqs (24)–(25), their practical usefulness is limited by the fact that they require a complete enumeration of set X. As already noticed, set X is finite, but it can also grow extremely large.

However, in the rest of this section we shall show that, in the considered context, X admits a distributed representation that enables the rewriting of the above LPs in a much more compact form in terms of the employed numbers of variables and constraints.

The alternative LP formulations of the considered "dual" problem: The starting point for developing the new LP formulations of the considered "dual" problem is the observation provided in the closing part of Section 3 that the constraints defining the set X are totally separable across the agents $a \in \mathcal{A}$. Hence, the vectors $\boldsymbol{x}$ that are the elements of set X can be perceived as the concatenation of some vectors $\boldsymbol{x}_a,\ a \in \mathcal{A}$, with each vector $\boldsymbol{x}_a$ living in a space X_a that is defined by the corresponding subset of the Constraints (2)–(6) and (10) that refer to agent a; more formally,

$$X = \times_{a \in \mathcal{A}} X_a \tag{35}$$

From a more conceptual standpoint, each set $X_a,\ a \in \mathcal{A}$, encodes all the possible routes in the guidepath network G that take agent a from initial location s_a to its destination location d_a within the provided time span T. A compact way to represent all these routes is by an acyclic digraph $\mathcal{G}_a$. The nodes of this digraph are labeled by (e, t) and signify the placement of agent a at (directed) edge e at time period t. On the other hand, the edges of $\mathcal{G}_a$ connect nodes (e, t) for $t \in \{0, 1, \ldots, T-1\}$ to nodes $(e', t+1)$ with $e' \in \{e\} \cup e^{\bullet}$. For T adequately large to ensure the feasibility of the original MIP of Section 2, the digraph $\mathcal{G}_a$ will have node $(s_a, 0)$ as its single "source" node, and node (d_a, T) as its single "sink" node. Hence, each feasible route in X_a corresponds to a path leading from node $(s_a, 0)$ to node (d_a, T). Furthermore, the connectivity of digraph $\mathcal{G}_a$ encodes the additional requirement that a feasible route for agent a does not leave the destination edge d_a once it has reached it for the first time; i.e., for every node $(d_a, t),\ t \in \mathcal{T} \setminus \{T\}$, the only emanating edge from this node is the edge leading to node $(d_a, t+1)$. Finally, it is clear that each digraph $\mathcal{G}_a,\ a \in \mathcal{A}$, can have no more than $|E| \cdot T$ nodes, and it can be constructed through elementary reachability analysis on the guidepath network G in time $O(|E|^2 \cdot T)$.

Next, we employ the graphical representation of the sets $X_a,\ a \in \mathcal{A}$, that was defined in the previous paragraph, in order to provide a more efficient encoding of Constraints (32) and (33) in the LP formulation of Eqs (31)–(34). A closer examination of these two constraints will reveal that they essentially employ the convex hull of the vector set $\{\boldsymbol{g}(\boldsymbol{x}) : \boldsymbol{x} \in X\}$, where the vector function $\boldsymbol{g}(\cdot)$ is the function that was introduced in Eq. (26). Let us denote this convex hull by $Conv(\{\boldsymbol{g}(\boldsymbol{x}) : \boldsymbol{x} \in X\})$. The next proposition establishes a distributed representation of $Conv(\{\boldsymbol{g}(\boldsymbol{x}) : \boldsymbol{x} \in X\})$ by means of the convex hulls of the vector sets $X_a,\ a \in \mathcal{A}$.

Proposition 2 *It holds that:*

$$Conv(\{\boldsymbol{g}(\boldsymbol{x}) : \boldsymbol{x} \in X\}) = \ \{\boldsymbol{g}(\boldsymbol{q}) : \boldsymbol{q} \in \times_{a\in\mathcal{A}} Conv(X_a)\} \tag{36}$$

Proof: As remarked in Section 3, each component of the vector function $\boldsymbol{g}(\boldsymbol{x})$ is a linear function of $\boldsymbol{x}$, and therefore,

$$\boldsymbol{g}(\boldsymbol{x}) \ = \ A \cdot \boldsymbol{x} + \boldsymbol{\beta}^2 \tag{37}$$

for an appropriately defined matrix A and vector $\boldsymbol{\beta}^2$. [9]

Furthermore, from Eqs (26), (37) and the definition of $Conv(\{\boldsymbol{g}(\boldsymbol{x}) : \boldsymbol{x} \in X\})$, we have:

$$\begin{aligned} Conv(\{\boldsymbol{g}(\boldsymbol{x}) : \boldsymbol{x} \in X\}) \ = \\ & \{g \equiv \sum_{\boldsymbol{x}\in X} \xi_{\boldsymbol{x}}(A \cdot \boldsymbol{x} + \boldsymbol{\beta}^2) :\ \forall \boldsymbol{x} \in X,\ \xi_{\boldsymbol{x}} \geq 0;\ \sum_{\boldsymbol{x}\in X} \xi_{\boldsymbol{x}} = 1.0\} = \\ & \{g \equiv A \cdot \sum_{\boldsymbol{x}\in X} \xi_{\boldsymbol{x}} \cdot \boldsymbol{x} + \boldsymbol{\beta}^2 :\ \forall \boldsymbol{x} \in X,\ \xi_{\boldsymbol{x}} \geq 0;\ \sum_{\boldsymbol{x}\in X} \xi_{\boldsymbol{x}} = 1.0\} = \\ & \{g \equiv A \cdot \boldsymbol{q} + \boldsymbol{\beta}^2 : \boldsymbol{q} \in Conv(X)\} \end{aligned} \tag{38}$$

Finally, from Eq. (35), we also have that

$$Conv(X) \ = \ \times_{a\in\mathcal{A}} Conv(X_a) \tag{39}$$

and the proof is complete. □

Each element $\boldsymbol{q}_a \in Conv(X_a),\ a \in \mathcal{A}$, can be represented by means of a flow $\boldsymbol{f}_a$ on the corresponding graph $\mathcal{G}_a$, that transfers a unit of fluid from the "source" node of $\mathcal{G}_a$ to its "sink" node. More specifically, for some $\boldsymbol{q}_a \in Conv(X_a),\ a \in \mathcal{A}$, let

$$\boldsymbol{q}_a = \sum_{\boldsymbol{x}_a\in X_a} \xi_{\boldsymbol{x}_a} \boldsymbol{x}_a;\ \forall \boldsymbol{x}_a \in X_a,\ \xi_{\boldsymbol{x}_a} \geq 0;\ \sum_{\boldsymbol{x}_a\in X_a} \xi_{\boldsymbol{x}_a} = 1.0 \tag{40}$$

Then, the flow $\boldsymbol{f}_a$ on the graph $\mathcal{G}_a$ that represents the vector $\boldsymbol{q}_a$, sends through some edge $((e,t),(e',t+1))$ of this graph an amount of fluid equal to the total weight of the vectors $\boldsymbol{x}_a$

[9] A complete definition of matrix A and vector $\boldsymbol{\beta}^2$ can be obtained from the parsing of the right-hand-side of Eq. (11) that defines the Lagrangian function employed in this work.

that involve the transition $\big((e,t),(e',t+1)\big)$ in the corresponding routes. The vector $\boldsymbol{q}_a$, itself, consists of the amounts of fluid that are routed through the different nodes (e,t) of the graph $\mathcal{G}_a$ by the aforementioned flow pattern for $\boldsymbol{f}_a$.

Hence, for each $a \in \mathcal{A}$, the set $Conv(X_a)$ can be represented parametrically by a set of linear equations

$$F_a \cdot \boldsymbol{f}_a = \boldsymbol{\beta}_a^1 \quad ; \quad \boldsymbol{f}_a \geq \mathbf{0} \tag{41}$$

$$\boldsymbol{q}_a = Q_a \cdot \boldsymbol{f}_a \tag{42}$$

with the matrices F_a, Q_a and the vecrtor β_a^1 suitably defined. More specifically, in the above representation, Eq. (41) expresses the "flow balance" equations that must be satisfied by vector $\boldsymbol{f}_a$; these equations are determined by the topology of the corresponding graph $\mathcal{G}_a$, and the unit volume of the transferred fluid. On the other hand, Eq. (42) defines the vector $\boldsymbol{q}_a$ as a linear function of $\boldsymbol{f}_a$, as described in the previous paragraphs.[10]

Furthermore, we combine the linear systems of equations that are defined in Eq. (41) for each agent $a \in \mathcal{A}$, into the single equation

$$F \cdot \boldsymbol{f} = \boldsymbol{\beta}^1 \quad ; \quad \boldsymbol{f} \geq \mathbf{0} \tag{43}$$

where F is a block-diagonal matrix that collects the matrices $F_a,\ a \in \mathcal{A}$, and $\boldsymbol{\beta}^1$ is the vector that results from the concatenation of the vectors $\boldsymbol{\beta}_a^1,\ a \in \mathcal{A}$. Also, we shall let Q denote the block-diagonal matrix with its diagonal blocks being the matrices $Q_a,\ a \in \mathcal{A}$, and define

$$\hat{A} \equiv A \cdot Q \tag{44}$$

Finally, by combining all the above developments, the original LP formulation of Eqs (31)–(34) can be rewritten as

$$\min_{\boldsymbol{q},\boldsymbol{z},\psi} \psi \tag{45}$$

s.t.

$$\begin{bmatrix} F & 0 & 0 \\ -\hat{A} & -I & \mathbf{1}_{\boldsymbol{\nu}} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{z} \\ \psi \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta}^1 \\ \boldsymbol{\beta}^2 \end{bmatrix} \tag{46}$$

$$\boldsymbol{f} \geq \mathbf{0}; \quad \boldsymbol{z} \geq \mathbf{0} \tag{47}$$

[10] A more exact characterization of the elements F_a, $\boldsymbol{\beta}_a^1$ and Q_a that appear in Eqs (41) and (42), is provided in Section 5.

Also, the dual of the above LP has the form:

$$\max_{\boldsymbol{\eta},\boldsymbol{\rho}} \ (\boldsymbol{\beta}^1)^T \cdot \boldsymbol{\eta} + (\boldsymbol{\beta}^2)^T \cdot \boldsymbol{\rho} \tag{48}$$

s.t.

$$F^T \cdot \boldsymbol{\eta} - \hat{A}^T \cdot \boldsymbol{\rho} \ \leq \ 0 \tag{49}$$

$$\mathbf{1}_{\boldsymbol{\nu}}^T \cdot \boldsymbol{\rho} \ = \ 1.0 \tag{50}$$

$$\boldsymbol{\rho} \geq \mathbf{0} \tag{51}$$

The vectors $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$ that constitute the decision variables in this last formulation, collect, respectively, the dual variables for the constraints that correspond to the first and the second rows in Eq. (46). Furthermore, the LP of Eqs (48)–(51) is the analogue of the original LP formulation of Eqs (27)–(30) in the distributed representation of the set X and its convex hull $Conv(X)$ that were introduced in the previous paragraphs. This analogy is further characterized and analyzed in the following theorem.

Theorem 1 *The LP formulation of Eqs (48)–(51) is a valid representation of the "dual" problem of Eqs (24)–(25) modulo the term* $T+1$ *that appears in the right-hand-side of Eq. (24). Furthermore, for any optimal solution of this LP,* $(\boldsymbol{\eta}^*, \boldsymbol{\rho}^*)$*, the vector* $\boldsymbol{\rho}^*$ *defines an optimal set of Lagrange multipliers for the "dual" problem of Eqs (24)–(25).*

Proof: First we notice that the two LP formulations of Eqs (48)–(51) and Eqs (27)–(30) must have the same optimal objective value, since (i) they are the respective duals to the two LPs that are defined by the equation sets (45)–(47) and (31)–(34), and (ii) the last two LPs are equivalent by construction. But the optimal value of the LP formulation of Eqs (27)–(30) is equal to the optimal value of the "dual" problem of Eqs (24)–(25) modulo the term $T+1$ that appears in the right-hand-side of Eq. (24). This proves the first part of Theorem 1.

To establish the second part of the theorem, first we notice that Proposition 2 implies that in the LP formulation of Eqs (27)–(30), Constraint (28) can be substituted by the constraint:

$$\forall \boldsymbol{q} \in conv(X), \ \ u \ \leq \ \boldsymbol{g}(\boldsymbol{q})^T \cdot (\boldsymbol{\lambda}^T, \boldsymbol{\mu}^T, \boldsymbol{\nu}^T)^T \tag{52}$$

Next, let us consider a "flow" vector $\boldsymbol{f}$ that satisfies the constraints of Eq. (43), and take the inner product of this vector $\boldsymbol{f}$ with the left-hand-side of Eq. (49); the non-negativity of the elements of $\boldsymbol{f}$, together with Eqs (42), (44) and Proposition 2, imply that

$$\forall \boldsymbol{f} \text{ s.t. } F \cdot \boldsymbol{f} = \boldsymbol{\beta}^1 \ ; \ \boldsymbol{f} \geq \mathbf{0} :$$
$$\boldsymbol{f}^T \cdot F^T \cdot \boldsymbol{\eta} - \boldsymbol{f}^T \cdot \hat{A}^T \cdot \boldsymbol{\rho} \ = (\boldsymbol{\beta}^1)^T \cdot \boldsymbol{\eta} + (\boldsymbol{\beta}^2)^T \cdot \boldsymbol{\rho} - \boldsymbol{g}(\boldsymbol{q})^T \cdot \boldsymbol{\rho} \leq \ 0 \quad (53)$$

where $\boldsymbol{q} = Q \cdot \boldsymbol{f} \in Conv(X)$.

From the above developments, it is clear that any feasible solution for the LP of Eqs (48)–(51) defines a feasible solution for the LP of Eqs (27)–(30) by setting $u \equiv (\boldsymbol{\beta}^1)^T \cdot \boldsymbol{\eta} + (\boldsymbol{\beta}^2)^T \cdot \boldsymbol{\rho}$ and $(\boldsymbol{\lambda}^T, \boldsymbol{\mu}^T, \boldsymbol{\nu}^T)^T \equiv \boldsymbol{\rho}$, and these two feasible solutions give the same objective value to the two LPs, namely, the value of $u \equiv (\boldsymbol{\beta}^1)^T \cdot \boldsymbol{\eta} + (\boldsymbol{\beta}^2)^T \cdot \boldsymbol{\rho}$. But then, the second part of Theorem 1 results by considering an optimal solution $(\boldsymbol{\eta}^*, \boldsymbol{\rho}^*)$ of the LP of Eqs (48)–(51). □

Some complexity considerations: It should be clear from the above discussion that each diagonal block $F_a, a \in \mathcal{A}$, in the sub-matrix F appearing in the left-hand-side of Eq. (46), has a dimensionality of $O(|E| \cdot T) \times O(|E|^2 \cdot T)$. At the same time, matrix $\hat{A}$ in the left-hand-side of Eq. (46) has a number of rows equal to $\dim(\boldsymbol{\lambda}^T, \boldsymbol{\mu}^T, \boldsymbol{\nu}^T)$ (i.e., the total number of the variables that appear in the "dual" problem), and a number of columns equal to $\dim(\boldsymbol{f})$, which is $O(|\mathcal{A}| \cdot |E|^2 \cdot T)$. Hence, the numbers of variables and constraints in the LP formulation of Eqs (45)–(47) are polynomially related to the primary parameters that define the original MIP formulation, i.e., (i) the number of edges, $|E|$, in the guidepath network G, (ii) the number of the circulating agents, $|\mathcal{A}|$, and employed time horizon T. And, of course, a similar remark applies to the numbers of variables and constraints for the dual LP of Eqs (48)–(51). Hence, the derived LP of Eqs (48)–(51) is, indeed, a very convenient representation of the original "dual" problem of Eqs (24)–(25).

In Section 6 we shall also present a series of numerical experiments that will demonstrate more concretely the ability of the LP formulation of Eqs (48)–(51) to provide tight lower bounds to the considered scheduling problem in a computationally robust and efficient manner. But before turning to these computational developments, in the next section we provide another derivation of this LP that provides further insights for its origin and its informational content.

5 An alternative MIP formulation of the considered traffic scheduling problem

In this section, we present an alternative MIP formulation for the traffic scheduling problem of Section 2. This formulation is motivated by the LP formulation of Eqs (45)–(47), and it provides the basis for an alternative interpretation of the results of Section 4. More specifically, the new MIP formulation of the considered scheduling problem that is presented in this section, takes the form of an *integral multi-commodity flow (IMCF)* problem [1], where the transported commodities are the traveling agents. This IMCF problem is formulated on a new acyclic digraph $\mathcal{G}$ that encodes all the potential routes for the traveling agents over the considered time horizon T, and it also includes a number of *"side" constraints* expressing the various restrictions that are imposed by the zone allocation protocol. Furthermore, the IMCF structure of the new MIP formulation gives it the *"integrality"* property of [12], and therefore, the lower bound to the optimal makespan that is obtained through Lagrangian duality theory from this MIP, is equal to the bound that is obtained from its LP relaxation.[11] The picture is completed by establishing that the LP of Eqs (45)–(47) is essentially the *LP relaxation* of the new MIP.

In the rest of this section, we develop all the results that were outlined in the previous paragraph. An additional gain from the following developments is a more explicit characterization of the various elements that appear in Eq. (46). Finally, as pointed out in the introductory section, the following results also establish some affinity between this work and the works of [39, 20] that have highlighted the existing connection between the considered class of traffic scheduling problems and the IMCF model.

The alternative MIP formulation of the traffic scheduling problem of Section 2: As already mentioned, the new formulation of the traffic scheduling problem that is considered in this work, perceives the route of each agent $a \in \mathcal{A}$ as an integral unitary flow $\boldsymbol{\phi}_a$. More specifically, flow $\boldsymbol{\phi}_a$ is defined on an acyclic digraph $\mathcal{G}$ that, itself, is defined as follows: The node set $V_{\mathcal{G}}$ of graph $\mathcal{G}$ is equal to $E \times T$, and its edge set, $E_{\mathcal{G}}$, consists of the nodal pairs $\big((e,t),(e',t+1)\big)$ with $e' \in e^{\bullet} \cup \{e\}$. Flow $\boldsymbol{\phi}_a$ carries a unit of fluid from node $(s_a, 0)$ to node (d_a, T). Hence, the flow vector $\boldsymbol{\phi}_a$ is of dimensionality $|E_{\mathcal{G}}|$, and it must satisfy the following equations:

$$\sum_{e' \in s_a^{\bullet} \cup \{s_a\}} \phi_a\big((s_a,0),(e',1)\big) = 1.0 \tag{54}$$

$$\sum_{e' \in {}^{\bullet}d_a \cup \{d_a\}} \phi_a\big((e',T-1),(d_a,T)\big) = 1.0 \tag{55}$$

$$\forall (e,t) \in V_{\mathcal{G}} \setminus \big\{(s_a,0),(d_a,T)\big\},$$
$$\sum_{e' \in {}^{\bullet}e \cup \{e\}} \phi_a\big((e',t-1),(e,t)\big) = \sum_{e' \in e^{\bullet} \cup \{e\}} \phi_a\big((e,t),(e',t+1)\big) \tag{56}$$

$$\forall \big((e,t),(e',t+1)\big) \in E_{\mathcal{G}}, \quad \phi_a\big((e,t),(e',t+1)\big) \in \{0,1\} \tag{57}$$

[11] As remarked in the introductory section, in the general case, the bounds obtained from the LP relaxation of a MIP formulation might not be as tight as the corresponding bounds that are obtained through Lagrangian duality theory [3].

It should be clear from the above definitions that any flow vector $\boldsymbol{\phi}_a$ satisfying Eqs (54)–(57) defines a feasible route that takes agent $a \in \mathcal{A}$ from its initial location s_a at time 0, to its destination location d_a by time period T. Similarly, any feasible route for agent a can be represented as a flow vector $\boldsymbol{\phi}_a$. Hence, any possible schedule for the considered traffic scheduling problem is represented as a *multi-commodity flow* on digraph $\mathcal{G}$.

However, there is a need for further constraints that will establish the validity of the traffic schedules that are defined by the flow vectors $\boldsymbol{\phi}_a,\ a \in \mathcal{A}$, w.r.t. the imposed zone allocation protocol. These additional constraints can be derived as follows:

First we notice that the *"state"* variables $\boldsymbol{x}_{a,e,t}$ that were employed by the MIP formulation of Section 2, can be expressed by means of the "flow" vectors $\boldsymbol{\phi}_a$ as follows:

$$\forall a \in \mathcal{A},\ \forall e \in E,\ \boldsymbol{x}_{a,e,0} = \sum_{e' \in e^{\bullet} \cup \{e\}} \phi_a\big((e,0),(e',1)\big) \tag{58}$$

$$\forall a \in \mathcal{A},\ \forall e \in E,\ \boldsymbol{x}_{a,e,T} = \sum_{e' \in\ ^{\bullet}e \cup \{e\}} \phi_a\big((e',T-1),(e,T)\big) \tag{59}$$

$$\begin{aligned} &\forall a \in \mathcal{A},\ \forall e \in E,\ \forall t \in T \setminus \{0,T\}, \\ &\qquad x_{a,e,t} \ = \sum_{e' \in\ ^{\bullet}e \cup \{e\}} \phi_a\big((e',t-1),(e,t)\big) = \sum_{e' \in e^{\bullet} \cup \{e\}} \phi_a\big((e,t),(e',t+1)\big) \end{aligned} \tag{60}$$

But once the "state" variables $\boldsymbol{x}_{a,e,t}$ of the initial MIP formulation of Section 2 have been retrieved through Eqs (58)–(60), the constraint set of the new MIP formulation can be completed by appending to it Constraints (5), (7) and (8) of that first MIP formulation. As in the original MIP formulation, these three constraint sets express, respectively, (i) the requirement that an agent does not leave its destination edge once it reaches it, (ii) the unitary capacity of each zone in the guidepath network G, and (iii) the requirement that an agent can advance to a zone e at period t only if this zone was free during period $t-1$.

To complete the definition of the new MIP formulation, we must also define its objective function. This function is readily defined in terms of the new primary decision variables $\boldsymbol{\phi}_a,\ a \in \mathcal{A}$, by defining the following cost functions $\mathcal{C}_a,\ a \in \mathcal{A}$, on the edge set $E_{\mathcal{G}}$ of digraph $\mathcal{G}$:

$$\forall a \in \mathcal{A},\ \forall \big((e,t),(e',t+1)\big) \in E_{\mathcal{G}}, \quad \mathcal{C}_a\big((e,t),(e',t+1)\big) \equiv \begin{cases} 0 & \text{if } e = e' = d_a; \\ 1 & \text{otherwise.} \end{cases} \tag{61}$$

Then, the makespan of any feasible schedule represented by a set of flows $\boldsymbol{\phi}_a,\ a \in \mathcal{A}$, is given by $\max_{a \in \mathcal{A}} \{\mathcal{C}_a^T \cdot \boldsymbol{\phi}_a\}$, and therefore, the objective of our traffic scheduling problem can be expressed by

$$\min \omega \tag{62}$$

where $\omega \equiv \max_{a\in\mathcal{A}} \left\{\mathcal{C}_a^T \cdot \boldsymbol{\phi}_a\right\}$. Finally, the resulting formulation can be linearized by eventually minimizing the auxiliary variable ω, under the additional constraints

$$\forall a \in \mathcal{A},\ \omega \ \geq \ \mathcal{C}_a^T \cdot \boldsymbol{\phi}_a \tag{63}$$

The next theorem summarizes all the above developments, providing a succinct characterization of the new MIP formulation for the considered traffic scheduling problem.

Theorem 2 *An alternative MIP formulation for the traffic scheduling problem of Section 2 is provided by the objective function of Eq. (62), and the constraint sets that are defined by: (i) Eqs (54)–(57); (ii) the Constraints (5), (7) and (8) of that original MIP formulation of Section 2, where the original variables $x_{a,e,t}$ have been substituted by the corresponding expressions of Eqs (58)–(60); and (iii) Eq. (63). The primary decision variables of this new MIP formulation are the vectors $\boldsymbol{\phi}_a,\ a \in \mathcal{A}$, with each vector $\boldsymbol{\phi}_a$ defining a unitary integral flow on the digraph $\mathcal{G}$ that was defined at the beginning of this section.* □

The connection between the LP relaxation of the MIP formulation of Theorem 2 and the LP of Eqs (45)–(47): In order to see the connection between the MIP formulation of Theorem 2 and the LP of Eqs (45)–(47), let us consider a slight variation of the new MIP, where the integral unitary flows $\boldsymbol{\phi}_a,\ a \in \mathcal{A}$, are defined on the corresponding subgraphs $\mathcal{G}_a$ of the original digraph $\mathcal{G}$, that were defined in Section 4. We remind the reader that these subgraphs enforce the additional requirement of Constraint (5) in original MIP formulation of Section 2; hence, the employment of these subgraphs for the specification of the corresponding flows $\boldsymbol{\phi}_a$ removes the need for the employment of Constraint (5) in the new MIP formulation.

Also, let us write the constraints of the new MIP that are induced by Constraints (7) and (8) of the original MIP formulation of Section 2, in the form

$$\hat{A}_{(7,8)} \cdot \boldsymbol{\phi} + \boldsymbol{\beta}^2_{(7,8)} \ \leq \ 0 \tag{64}$$

where the vector $\boldsymbol{\phi}$ is the concatenation of the vectors $\boldsymbol{\phi}_a,\ a \in \mathcal{A}$, and the elements $\hat{A}_{(7,8)}$ and $\boldsymbol{\beta}^2_{(7,8)}$, that appear in the above equation, are defined accordingly. Furthermore, we replace the constraint of Eq. (63) with the constraint

$$\hat{A}_{(65)} \cdot \boldsymbol{\phi} - (T+1) \cdot \mathbf{1} \ \leq \ \omega \cdot \mathbf{1} \tag{65}$$

where the matrix $\hat{A}_{(65)}$ is appropriately defined.

Finally, Constraints (64) and (65) can be combined to the following constraint:

$$\begin{bmatrix} \hat{A}_{(7,8)} \\ \hat{A}_{(65)} \end{bmatrix} \cdot \boldsymbol{\phi} \;+\; \begin{bmatrix} \boldsymbol{\beta}^2_{(7,8)} \\ -(T+1) \cdot \mathbf{1} \end{bmatrix} \;\leq\; \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} \cdot \omega \tag{66}$$

Then, it is not hard to see that the LP of Eqs (45)–(47) is the LP relaxation of the MIP that is obtained from the MIP formulation of Theorem 2 through the slight modifications and the additional notation that were introduced in the previous paragraphs; we invite the reader to work out the relevant details.

The "integrality" property for the MIP formulation of Theorem 2: In the MIP formulation that was defined in the last paragraph, the "coupling" constraints across the different agents are the constraints that appear in Eq. (66). The relaxed version of our traffic scheduling problem that is obtained through the Lagrangian relaxation of these constraints, boils down to the solution of a set of "shortest path" problems that are defined on the corresponding digraphs $\mathcal{G}_a$ and possess a cost structure that depends on the employed Lagrange multipliers. This particular structure of the Lagrangian relaxation further implies that it will have an integral optimal solution, ϕ^*, for any selection of Lagrange multipliers; i.e., it possesses the "integrality" property of [12]. But then, Theorem 2 in [12] implies that this Lagrangian relaxation cannot improve any further the performance bound that is obtained from the LP relaxation of the corresponding MIP, and that the bounds provided by the LP formulations of Section 4 remain the tightest possible that can be obtained through the presented analysis.

Finally, we also notice, for completeness, that the bound provided by the LP relaxation of the MIP formulation of Section 2 might not be as tight as the bound that is obtained through the corresponding "dual" problem of Eqs (24)–(25); a counter-example establishing this fact is provided in [5].

6 A numerical experiment

The performed experiment: In this section, we demonstrate and empirically assess, by means of a numerical experiment, (i) the computational tractability of the LP formulation of Eqs (48)–(51), and (ii) the tightness of the obtained bounds. More specifically, in the presented experiment, we formulated and solved the LP of Eqs (48)–(51) for a number of instantiations of the traffic scheduling problem considered in Section 2, that were defined by means of the guidepath network depicted in Figure 1.

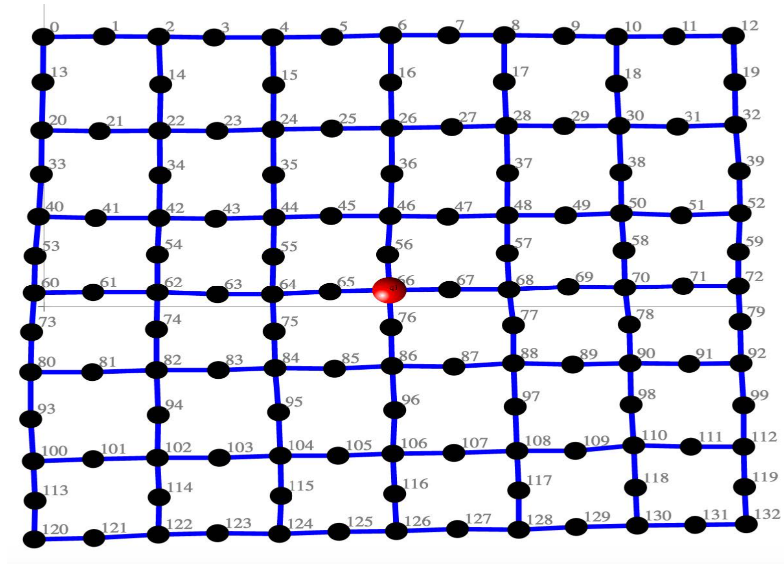

Figure 1: The guidepath network used in the numerical experiment that is presented in Section 6.

The guidepath network of Figure 1 provides 133 distinct zones for the traveling agents, organized in the depicted grid.[12] In the presented experiment, we generated randomly a number of instances of the original MIP formulation of Section 2, while varying the number of traveling agents from 3 to 30, with a step-increase of 3. For each of these levels, we generated five replications, and for each replication, we obtained an upper bound T to the optimal makespan w^* using some of the heuristic algorithms that are reported in [9, 7]. Subsequently, we formulated and solved the corresponding LP formulation of Eqs (48)–(51). All the LP formulations were solved through CPLEX, while the preparation of the input files for CPLEX from the original problem data was performed through MATLAB. The corresponding computation was executed on a 2013 Macbook Pro with a 2.4 GHz Intel Core i5 processor and 8 GB of 1600 MHz DDR3 RAM.

An empirical assessment of the computational tractability of the LP formulation of Eqs (48)–(51): Figure 2 plots the computational times required for setting up and solving the LP formulation of Eqs (48)–(51), as a function of the number of traveling agents; more specifically, the reported numbers are the averages of the computational times that were observed for the five corresponding replications. As it can be seen from the plot of Figure 2, the required computational times increase with the number of agents involved and the resulting congestion in the underlying guidepath network. But the presented approach remains tractable for pretty large instances of the underlying scheduling problem.

[12] In the graph of Figure 1 the available zones are encoded by the graph nodes and not by its edges; but the translation of this structure to the corresponding model of Section 2 is pretty straightforward.

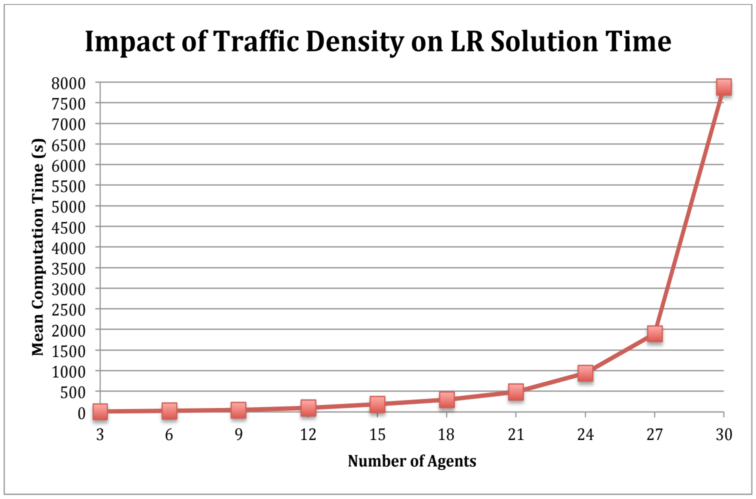

Figure 2: A plot reporting the computational times observed in the numerical experiment of Section 6.

In order to provide (i) a more vivid characterization of the congestion involved in the most challenging instantiations of the presented experiment, and (ii) an appreciation of the impact of this congestion on the observed computational times, we notice that a circulation of 30 agents in the depicted grid of Fig. 1 results in a node occupancy of this grid of almost 25%; i.e., one out of four nodes of this grid is occupied by some traveling agent. In the context of the considered experiment, this high congestion subsequently results in an increased sub-optimality of the solutions that are returned by our heuristic algorithms as estimates of the optimal makespan, and this fact further translates into inflated T-values for the LP formulation of Eqs (48)–(51). Finally, according to the complexity analysis that is provided at the end of Section 4, these inflated T-values increase the "size" of the LP formulation of Eqs (48)–(51), in terms of the numbers of variables and constraints that are employed in it.

Another pertinent remark for a better appreciation of the data that is provided in the plot of Fig. 2, is that the largest part of the times reported in Figure 2 was consumed by MATLAB for setting up the corresponding formulations. We believe that the reported times can be curtailed considerably by using a more streamlined code for this task, developed in a more basic programming language like C.

Finally, in order to provide the reader with some more concrete appreciation of the computational effort that was involved in the presented experiment, we also notice that the largest LP formulated and solved in this experiment employed 184,444 variables and 316,025 constraints.

An empirical assessment of the tightness of the bounds that are returned by the LP formulation of Eqs (48)–(51): Table 1 provides an empirical assessment of the tightness of the bounds that

Table 1: Assessing the quality of the obtained bounds for the problem instances of the experiment of Section 6.

# agents	D_1	H_1	D_2	H_2	D_3	H_3	D_4	H_4	D_5	H_5	Av. Rel. Gap
3	14	14	13	13	19	19	12	12	17	17	0
6	15	15	13	13	19	19	12	12	17	17	0
9	16	16	13	13	19	19	17	17	17	17	0
12	16	16	13	13	19	19	21	21	17	17	0
15	17	18	13	14	19	19	21	21	17	20	0.06
18	17	18	13	15	20	20	21	21	17	18	0.05
21	17	18	16	16	20	20	21	21	17	19	0.03
24	17	18	16	21	20	20	21	21	17	19	0.08
27	21	21	16	20	20	20	21	21	17	21	0.08
30	22	30	16	20	20	20	21	26	17	21	0.17

are obtained through the LP formulation of Eqs (48)–(51). Since it is not possible to compare these bounds against the optimal objective values of the corresponding MIPs, due to practical difficulties with solving these MIPs to optimality, we compare them against the makespan of the optimized schedules that are obtained through the heuristic schedulers that have been developed in [9]. More specifically, each line in Table 1 concerns the five problem instances that have been generated in the considered experiment for the number of agents that are reported in the first entry of the line, and for each of these five instances, it reports (i) the obtained bound in column D_i, and (ii) the makespan that is attained by the heuristic schedulers of [9], in column H_i, for $i = 1, \ldots, 5$. Furthermore, since the optimal solution for the MIP of Section 2 has an integer value, columns D_i of Table 1 report the ceilings of the optimal values of the corresponding LPs; i.e., these last values have been rounded up to the next integer. Finally, the rightmost entry of each line in Table 1 reports the "average relative gap" across the corresponding five problem instances, where the "relative gap" for a single instance i is computed by

$$\frac{H_i - D_i}{H_i}.$$

The coincidence of the entries in many of the pairs (D_i, H_i), in the data of Table 1, testifies to the high quality of, both, (i) the lower bounds that are obtained through the methodology that has been developed in this paper, and also (ii) the schedules that are returned by the heuristic scheduler of [9]. Furthermore, whenever these two entries differ for a certain pair, it is unclear whether the discrepancy is due to the poor quality of the bound D_i or the sub-optimality of the schedule that provides the corresponding value H_i.[13] But even with these inflations, the

[13]In fact, in certain cases, we can actually infer that the observed gap is due primarily to the sub-optimality of the schedule that is utilized in the estimation of the corresponding gap. As a concrete example, we refer to the case of pair (D_5, H_5) in the line of Table 1 corresponding to 15 agents. Looking at the column for H_5, we observe the sequence $\langle 17, 20, 18 \rangle$ for the rows corresponding to number of agents 12, 15 and 18. But the construction of

relative gaps that are reported in the last column of Table 1 are still quite low. Finally, we should also point out that the higher values that are observed in the last few rows of Table 1, can be explained by the remarks that were provided in the earlier parts of this section, regarding the challenges that are experienced by the schedulers of [9] as the density of the traveling agents in the underlying grid becomes pretty large; these remarks also explain the common trends that are observed in the last column of Table 1 and in the plot of Fig. 2.

7 Conclusions

This work has sought to develop (lower) performance bounds for a traffic scheduling problem that arises in many application contexts, ranging from industrial material handling and robotics to computer game animations and quantum computing. In a first approach, the sought bounds were obtained by applying the Lagrangian relaxation method to a MIP formulation of the considered scheduling problem that is based on a natural notion of "state" for the underlying traffic system and an analytical characterization of all the possible trajectories of this state over a predefined time horizon. But it was also shown that the corresponding "dual" problem that provides these bounds, can be transformed to a linear program (LP) with numbers of variables and constraints polynomially related to the size of the underlying traffic system and the employed time horizon in the MIP formulation. Furthermore, the derived LP formulation constitutes the LP relaxation of a second MIP formulation for the considered scheduling problem that can be obtained through an existing connection between this problem and the IMCF model. Finally, the theoretical developments of the paper were complemented with a computational part that demonstrates the efficacy of the pursued methods in terms of the quality of the derived bounds, and their computational tractability.

Our future work will seek to further assess the potential of the analytical insights and of the computational capability that were established in this paper, towards the development of further analytical methodology for the computation of near-optimal solutions for the considered traffic scheduling problems. In fact, it is also possible to extend the applicability of the presented formulations and bounding methods to other, more general problems that involve sequential resource allocation, like various versions of the notorious "job shop" scheduling problem [25]; the systematic exploration of this possibility, and of the potential gains incurred by it, is another part of our future work. Finally, at a more general level, our future work will also seek the further development of the MPC control scheme for the considered traffic systems that has been

the corresponding problem instances through the addition of three agents from each instance to the next, implies that the optimal makespans for these three problem instances should be monotonically increasing. Therefore, the actual optimal makespan for the fifth problem instance with 15 traveling agents is actually 17 or 18, and not 20.

presented in [9, 26], so that it provides a complete control framework for the broadest possible range of these traffic systems.

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin. *Network Flows: Theory, Algorithms and Applications*. Prentice Hall, Englewood Cliffs, NJ, 1993.

[2] V. Auletta, A. Monti, M. Parente, and P. Persiano. A linear-time algorithm for the feasibility of pebble motion on trees. *Algorithmica*, 23:223–245, 1999.

[3] D. P. Bertsekas. *Nonlinear Programming (2nd ed.)*. Athena Scientific, Belmont, MA, 1999.

[4] C. G. Cassandras and S. Lafortune. *Introduction to Discrete Event Systems (2nd ed.)*. Springer, NY,NY, 2008.

[5] G. Daugherty. *Multi-Agent Routing in Shared Guidepath Networks*. PhD thesis, Georgia Tech, Atlanta, GA, 2017.

[6] G. Daugherty, S. Reveliotis, and G. Mohler. Some novel traffic coordination problems and their analytical study based on Lagrangian Duality theory. In *Proceedings of the 55th IEEE Conf. on Decision and Control (CDC 2016)*, pages –. IEEE, 2016.

[7] G. Daugherty, S. Reveliotis, and G. Mohler. Optimized multi-agent routing in guidepath networks. In *Proceedings of the 2017 IFAC World Congress*, pages –. IFAC, 2017.

[8] G. Daugherty, S. Reveliotis, and G. Mohler. Solving the Lagrangian dual problem for some traffic coordination problems through linear programming. In *Proceedings of the 56th IEEE Conf. on Decision and Control (CDC 2017)*, pages –. IEEE, 2017.

[9] G. Daugherty, S. Reveliotis, and G Mohler. Optimized multi-agent routing for a class of guidepath-based transport systems. *IEEE Trans. on Automation Science and Engineering*, 16:363–381, 2019.

[10] G. Desaulniers, A. Langevin, D. Riopel, and B. Villeneuve. Dispatching and conflict-free routing of automated guided vehicles: An exact approach. *The Intl. Jrnl of Flexible Manufacturing Systems*, 15:309–331, 2003.

[11] M. L. Fisher. The Lagrangian relaxation method for solving integer programming problems. *Management Science*, 27:1–18, 1981.

[12] A. M. Geoffrion. Lagrangian relaxation for integer programming. *Math. Programming Studies*, 2:82–114, 1974.

[13] S. S. Heragu. *Facilities Design (3rd ed.)*. CRC Press, 2008.

[14] D. J. Hoitomt, P. B. Luh, and K. R. Pattipati. A practical approach to job-shop scheduling problems. *IEEE Trans. on Robotics & Automation*, 9:1–13, 1993.

[15] J. Huang, U. S. Palekar, and S. G. Kapoor. A labeling algorithm for the navigation of automated guided vehicles. *Journ. of Eng. for Industry*, 115:315–321, 1993.

[16] D. Kornhauser, G. Miller, and P. Spirakis. Coordinating pebble motion on graphs, the diameter of permutation groups, and applications. In *Proc. IEEE Symp. Found. Comput. Sci.*, pages 241–250. IEEE, 1984.

[17] B. Kouvaritakis and M. Cannon. *Model Predictive Control: Classical, Robust and Stochastic*. Springer, London, UK, 2015.

[18] N. N. Krishnamurthy, R. Batta, and M. H. Karwan. Developing conflict-free routes for automated guided vehicles. *Oper. Res.*, 41:1077–1090, 1993.

[19] D. G. Luenberger and Y. Ye. *Linear and Nonlinear Programming (4th ed.)*. Springer, NY, NY, 2016.

[20] H. Ma, C. Tovey, G. Sharon, S. Kumar, and S. Koenig. Multi-agent path finding with payload transfers and the package-exchange robot-routing problem. In *AAAI 2016*, pages 3166–3173, 2016.

[21] T. E. Morton and D. Pentico. *Heuristic Scheduling Systems*. John Wiley, NY, NY, 1993.

[22] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge, UK, 2010.

[23] T. Nishi and R. Maeno. Petri net decomposition approach to optimization of route planning problem for AGV systems. *IEEE Trans. on Automation Science and Engineering*, 7:523–537, 2010.

[24] C. H. Papadimitriou and K. Steiglitz. *Combinatorial Optimization: Algorithms and Complexity*. Dover, Mineola, NY, 1998.

[25] M. Pinedo. *Scheduling*. Prentice Hall, Upper Saddle River, NJ, 2002.

[26] S. Reveliotis. Preservation of traffic liveness in MPC schemes for guidepath-based transport systems. In *Proceedings of IEEE CASE 2018*, pages –. IEEE, 2018.

[27] S. Reveliotis and E. Roszkowska. On the complexity of maximally permissive deadlock avoidance in multi-vehicle traffic systems. *IEEE Trans. on Automatic Control*, 55:1646–1651, 2010.

[28] S. A. Reveliotis. Conflict resolution in AGV systems. *IIE Trans.*, 32(7):647–659, 2000.

[29] Q. Sajid, R. Luna, and K. E. Bekris. Multi-agent path finding with simultaneous execution of single-agent primitives. In *5th Symposium on Combinatorial Search*, 2012.

[30] G. Sharon, R. Stern, A. Felner, and N. R. Sturtevant. Conflict-based search for optimal multi-agent pathfinding. *Artificial Intelligence*, 219:40–66, 2015.

[31] T. Standley. Finding optimal solutions to cooperative pathfinding problems. In *Proc. AAAI 2010*, 2010.

[32] T. Standley and R. Korf. Complete algorithms for cooperative pathfinding problems. In *Proc. 22nd Intl. Joint Conf. Artif. Intell.*, 2011.

[33] P. Surynek. Towards optimal cooperative path planning in hard setups through satisfiability solving. In *Proc. 12th Pacific Rim Int. Conf. Artif. Intell.*, pages 564–576, 2012.

[34] G. Wagner and H. Choset. Subdimensional expansion for multirobot path planning. *Artificial Intelligence*, 219:1–24, 2015.

[35] K-H. C. Wang and A. Botea. MAPP: A scalable multi-agent path planning algorithm with tractability and completeness guarantees. *Journal of Artificial Intelligence Research*, 42:55–90, 2011.

[36] R. M. Wilson. Graph puzzles, homotopy, and the alternating group. *Journal of Combinatorial Theory, B*, 16:86–96, 1974.

[37] L. A. Wolsey. *Integer Programming.* John Wiley & Sons, NY, NY, 1998.

[38] J. Yu and S. M. LaValle. Optimal multi-robot path planning on graphs: Structure and computational complexity. *ArXiv: http://arxiv.org/pdf/1507.03289v1.pdf*, 2015.

[39] J. Yu and S. M. LaValle. Optimal multirobot path planning on graphs: Complete algorithms and effective heuristics. *IEEE Trans. on Robotics*, 32:1163–1177, 2016.

[40] J. Yu and D. Rus. Pebble motion on graphs with rotations: Efficient feasibility tests and planning algorithms. In *Algorithmic Foundations of Robotics XI*, 2015.