ISYE 7201: Production & Service Systems
Spring 2023
Instructor: Spyros Reveliotis
2nd Midterm Exam (Take Home)
Release Date: March 1, 2023
Due Date: March 5, 2023

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

Please, send me your responses as a pdf file attached to an email. Name the pdf file by your last name (only). The pdf file can be a scan or photos of a hand-written document, but, please, write your answers clearly and thoroughly. Also, make sure that the pdf file is not too big; you can reduce the size of your file by loading it into Adobe Acrobat and saving it with the "reduced" size option before emailing it to me.

Finally, report any external sources (other than your textbook) that you referred to while preparing the solutions.

Problem 1 (20 points): Arrivals occur according to a Poisson process with rate λ. If there are n arrivals in the interval $[0, T]$, compute the expected arrival time of the last arrival.

Problem 2 (20 pts): Suppose that an one-cell organism can be in one of two states: either A or B. An individual in state A will change to state B at an exponential rate α. An individual in state B divides into two new individuals of type A at an exponential rate β. At time $t = 0$, we have only one individual from this organism in state A.

i. (10 pts) Define an appropriate Continuous-Time Markov Chain (CTMC) that traces the evolution of the population that will be generated by this cell over time. For this Markov chain, define carefully (i) the state space, (ii) the instantaneous rates of the exponential distributions that determine the sojourn times of the various states, and (iii) the one-step transition probability distribution that governs the transition out of each state.

ii. (5 pts) Draw the State Transition Diagram (STD) of the CTMC defined in part (i).

iii. (5 pts) Is the above CTMC ergodic?

Problem 3 (20 points): A small barbershop, operated by a single barber, has room for at most three customers. Potential customers arrive according to a Poisson process with rate three customers per hour, and the customer service times are normally distributed with a mean of 15 minutes and st. deviation of 7 minutes. Arriving customers who find the shop full just leave.

i. (5 pts) Model the operation of this barbershop as a Continuous-Time Markov Chain (CTMC) by approximating the service-time distribution with a 3-stage Erlang.

ii. (5 pts) What is the average number of customers in this barbershop at steady-state?

iii. (5 pts) What is the percentage of arriving customers that will actually receive service, when the barbershop operates in steady-state?

iv. (5 pts) What is the expected total time in the barbershop for a customer who enters the barbershop under operation in steady-state?

Please, justify thoroughly all your answers and demonstrate clearly all your computations.

Problem 4 (20 points): Consider the CTMC that models the dynamics of the M/M/1 queue in page 80 of the Primer.

i. (10 pts) Uniformize this CTMC using the rate $v = \lambda + \mu$ and use the embedded DTMC of the uniformized CTMC to show that a necessary condition for the ergodicity of the original CTMC is

$$\lambda < \mu$$

ii. (10 pts) Argue that the condition of part (i) is also sufficient.

Problem 5 (20 points): A shuttle moves among three locations. From location 1, the shuttle is directed half of the time towards location 2 and the remaining half towards location 3. From location 2, the shuttle is directed 1/3 of the time towards location 1 and 2/3 of the time towards location 3. From location 3 the shuttle is always directed to location 1. The mean travel times between the different locations are as follows: $t_{1,2} = 20\ min$, $t_{1,3} = 30\ min$ and $t_{2,3} = 30\ min$. Every time that it reaches a location, the shuttle departs immediately for its next location.

i. (10 pts) What is the limiting probability that the shuttle's most recent stop was location $i,\ i = 1, 2, 3$?

ii. (5pts) What is the limiting probability that the shuttle is heading to location 2?

iii. (5 pts) What fraction of time is the shuttle traveling from 2 to 3?

Problem 1

Let r.v. $T^{(n)}$ denote the time of the n-th arrival, and $F^{(n)}(t)$ and $f^{(n)}(t)$ the corresponding cdf and pdf.

Then

$$\forall t \in [0, T], \quad F^{(n)}(t) = P\left[T^{(n)} \leq t \mid N(T) = n\right] =$$

$$= P\left[N(t) = n \wedge N(T-t) = 0\right] / P\left[N(T) = n\right] =$$

$$= e^{-\lambda t} \frac{(\lambda t)^n}{n!} e^{-\lambda(T-t)} \frac{(\lambda(T-t))^0}{0!} \Big/ e^{-\lambda T} \frac{(\lambda T)^n}{n!} =$$

$$= e^{-\lambda T} (\lambda t)^n \Big/ e^{-\lambda T} (\lambda T)^n = \left(\frac{t}{T}\right)^n \qquad (1)$$

Also,

$$\forall t \in [0, T], \quad f^{(n)}(t) = \frac{dF^{(n)}(t)}{dt} = \frac{1}{T^n} n t^{n-1} \qquad (2)$$

<u>Remark:</u> For "sanity checking", notice also that (2) implies that

$$\int_0^T f^{(n)}(t) dt = \int_0^T \frac{1}{T^n} n t^{n-1} dt =$$

$$= \frac{1}{T^n} \left[t^n\right]_0^T = \frac{1}{T^n} T^n = 1.$$

Finally,

$$E[T^{(n)}] = \int_0^T t f^{(n)}(t)\, dt =$$

$$= \frac{n}{T^n} \int_0^T t^n dt = \frac{n}{n+1} \frac{1}{T^n} T^{n+1} =$$

$$= \frac{n}{n+1} T. \qquad (3)$$

<u>Remark:</u> for $n = 1$, (3) implies that

$E[T^{(1)}] = \frac{1}{2} T$, which is consistent with the uniform distribution of $T^{(1)}$ that was established in class.

Problem 2

(i, ii) The state of this stochastic process can be modeled as $\underline{x}(t) = (x_A(t), x_B(t))^T$, where

$x_A(t)$ = number of type-A individuals in the population at time t

$x_B(t)$ = number of type-B individuals in the population at time t.

Let

$(n_A, n_B)^T$ denote a state of this stochastic process. Then, the transition out of this state constitutes an exponential race for the transformation of one member of the corresponding population of $n_A + n_B$ members. Hence, the total transition rate out from this state is

$$U(n_A, n_B) = n_A \cdot a + n_B \cdot b \quad (1)$$

Furthermore, the resulting state is

* $(n_A - 1, n_B + 1)$ w.p. $n_A \cdot a / (n_A \cdot a + n_B \cdot b)$ (2)

* $(n_A + 2, n_B - 1)$ w.p. $n_B \cdot b / (n_A \cdot a + n_B \cdot b)$ (3)

Hence, the considered process is a CTMC with its parameters defined by (1) - (3). Also,

$$\underline{x}_0 = \underline{x}(0) = (1, 0)^T .$$

The STD of this CTMC is as follows

$$\begin{tikzcd}
 & & {[1 \ \ 0]^T} \arrow[d, "a"] & \\
 & & {[0 \ \ 1]} \arrow[d, "b"] & \\
 & & {[2 \ \ 0]} \arrow[d, "2a"] & \\
 & & {[1 \ \ 1]} \arrow[dl, "a"'] \arrow[dr, "b"] & \\
 & {[0 \ \ 2]} \arrow[dr, "2b"] & & {[3 \ \ 0]} \arrow[dl, "3a"'] \\
 & & {[2 \ \ 1]} \arrow[dl, "2a"'] \arrow[dr, "b"] & \\
 & {[1 \ \ 2]} \arrow[dl, "a"'] \arrow[dr, "2b"] & & {[4 \ \ 0]} \arrow[dl, "4a"'] \\
{[0 \ \ 3]} \arrow[dr, "3b"] & & {[3 \ \ 1]} \arrow[dl, "3a"'] \arrow[dr, "b"] & \\
 & {[2 \ \ 2]} \arrow[dl, "2a"'] \arrow[dr, "2b"] & & {[5 \ \ 0]} \arrow[d, "5a"'] \\
 {} & & {} & {}
\end{tikzcd}$$

(iii) It is clear from Eqs (2)-(3) and the above STD that this CTMC drifts continuously through states with increasing values of $n_A + n_B$, and therefore, it is not irreducible. Hence, it cannot be ergodic.

Problem 3

(i) We are asked to approximate the normal distr. $N(15, 7^2)$ with an Erlang$(3, \mu)$, where μ is the rate of the Exponential distr. associated with each of the three stages.

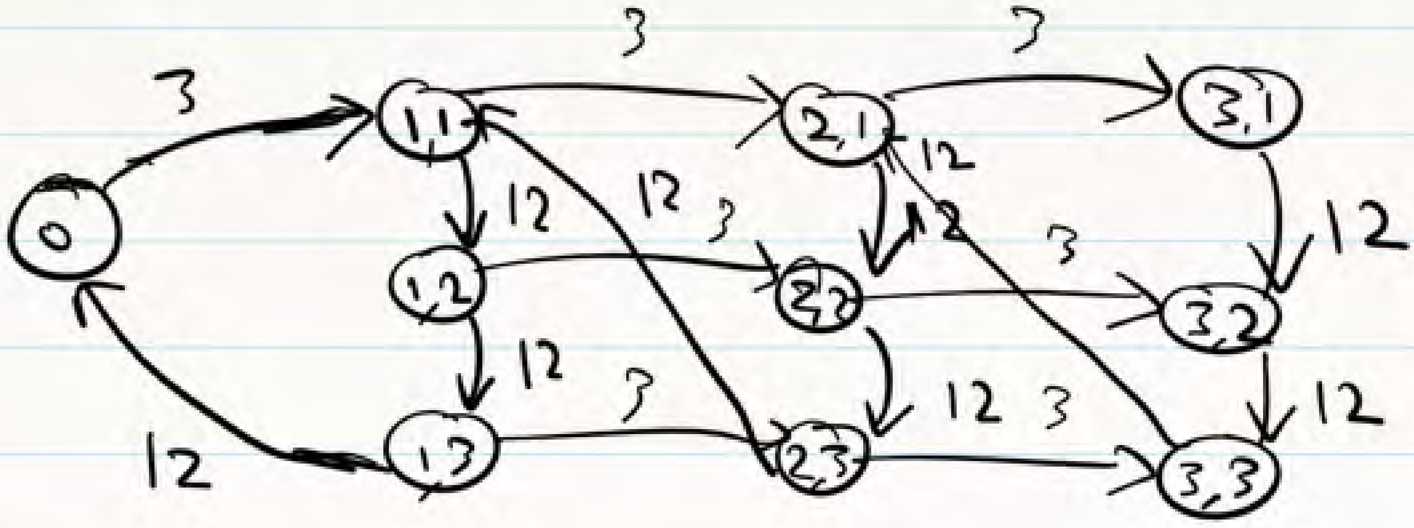
Let X denote the original r.v. and $\tilde{X}$ its approximation. Then, we must have:

$$E[X] = E[\tilde{X}] \Rightarrow$$

$$\Rightarrow 15\,\text{min} = 3/\mu \Rightarrow \mu = \frac{1}{5}\,\text{min}^{-1}$$

$$= 60/5\ \text{hr}^{-1} = 12\ \text{hr}^{-1}$$

The STD of the CTMC modeling the barbershop operation under the considered approximation is as follows:



In the above STD, state (i,j), $i=1,2,3$, $j=1,2,3$, implies that there are i customers in the barbershop, and the currently served customer is in the j-th stage of the Erlang distr. that models his service. Also the unit of the reported rates is hr^{-1}.

The infinitesimal generator of this CTMC is:

$$
R = \begin{array}{c} \\ 0 \\ 11 \\ 12 \\ 13 \\ 21 \\ 22 \\ 23 \\ 31 \\ 32 \\ 33 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 0 & 11 & 12 & 13 & 21 & 22 & 23 & 31 & 32 & 33 \end{array} \\
\left[\begin{array}{cccccccccc}
-3 & 3 & & & & & & & & \\
 & -15 & 12 & & 3 & & & & & \\
 & & -15 & 12 & & 3 & & & & \\
12 & & & -15 & & & 3 & & & \\
 & & & & -15 & 12 & & 3 & & \\
 & & & & & -15 & 12 & & 3 & \\
 & 12 & & & & & -15 & & & 3 \\
 & & & & & & & -12 & 12 & \\
 & & & & & & & & -12 & 12 \\
 & & & & 12 & & & & & -12
\end{array}\right]
\end{array}
$$

Since the embedded DTMC is irreducible and finite-state, the considered CTMC has a limiting distribution $\underline{p}$ that is obtained by solving

$$\begin{cases} \underline{p} \cdot R = 0 \\ \underline{p} \cdot \underline{1} = 1 \end{cases}$$

where $\underline{1}$ is the column vector with all its components equal to 1.0.
From this system of equations we get:

$$\begin{cases} p_0 = 0.3353 ; \\ p_{11} = 0.1310 ; \quad p_{12} = 0.1048 ; \quad p_{13} = 0.0838 \\ p_{21} = 0.0725 ; \quad p_{22} = 0.0789 ; \quad p_{23} = 0.0799 \\ p_{31} = 0.0181 ; \quad p_{32} = 0.0378 ; \quad p_{33} = 0.0578 \end{cases}$$

(ii) This number is

$$0 \cdot p_0 + 1 \cdot (p_{11} + p_{12} + p_{13}) + 2(p_{21} + p_{22} + p_{23}) + \\ + 3(p_{31} + p_{32} + p_{33}) = \\ = 1 \cdot 0.3196 + 2 \cdot 0.2313 + 3 \cdot 0.1137 = 1.1233$$

(iii) This is equal to the probability that an arriving customer will not find the system full. This probability is equal to

$$1-(P_{31}+P_{32}+P_{33}) = 1-0.1137 = 0.8863.$$

<u>Remark</u>: A more thorough treatment of this question needs a result that is known as PASTA (Poisson Arrivals See Time Averages). We shall discuss this result in the coming lectures.

(iv) This time can be computed as follows:

$$\{P_0 \cdot 15 + P_{11}(15+15) + P_{12}(10+15) + P_{13}(5+15) + P_{21}(15+15+15) + P_{22}(10+15+15) + P_{23}(5+15+15)\} \,/\, (P_0+P_{11}+P_{12}+P_{13}+P_{21}+P_{22}+P_{23}) =$$

$$= \frac{1}{0.8863}\{0.3353\times 15 + 0.1310\times 30 + 0.1048\times 25 + 0.0838\times 20 + 0.0725\times 45 + 0.0789\times 40 + 0.0799\times 35\} = 25.3532 \text{ min}$$

The above computation computes the expected time in system for an entering customer by considering the expected time in system for arrivals that occur at each of the nonblocking states of the underlying CTMC.

The division with $(p_0 + p_{11} + \cdots + p_{23})$ turns each probability p_i appearing in the computation to the conditional prob. for the corresponding state that accounts for the fact that the encountered state was nonblocking.

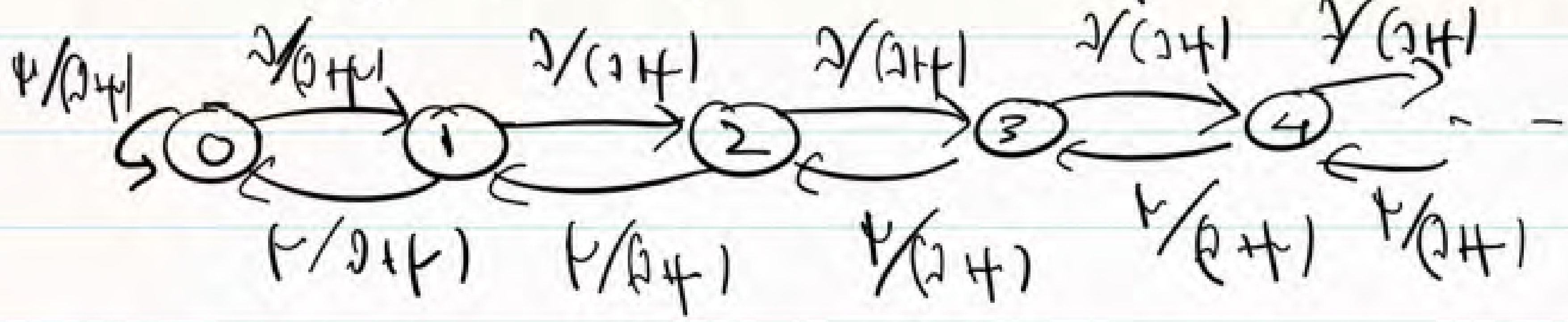
Also, the computation of the expected time in system for each encountered state employs the memoryless property of the exp. distr. that is associated with the various proc. stages (i, j).

Furthermore, a thorough justification of this computation requires PASTA.

Finally, this question can be answered in a simpler manner using Little's law, as we shall see in the discussion of queueing systems.

Problem 4

(i) The STD of the embedded DTMC of the uniformized process is as follows:



This DTMC is irreducible, and it will be positive recurrent if and only if it has a stationary distribution

$$\pi = (\pi_0, \pi_1, \pi_2, \ldots)$$

This distribution must satisfy the following flow balance equations:

$$\pi_0 \frac{\lambda}{\lambda+\mu} = \pi_1 \frac{\mu}{\lambda+\mu} \Rightarrow \pi_1 = \pi_0 \frac{\lambda}{\mu}$$

$$\pi_1 \frac{\lambda}{\lambda+\mu} = \pi_2 \frac{\mu}{\lambda+\mu} \Rightarrow \pi_2 = \pi_1 \frac{\lambda}{\mu} = \pi_0 \left(\frac{\lambda}{\mu}\right)^2$$

$$\pi_2 \frac{\lambda}{\lambda+\mu} = \pi_3 \frac{\mu}{\lambda+\mu} \Rightarrow \pi_3 = \pi_2 \frac{\lambda}{\mu} = \pi_0 \left(\frac{\lambda}{\mu}\right)^3$$

and more generally,

$\forall i = 0, 1, 2, \ldots$

$$\pi_i = \pi_0 \left(\lambda/\mu\right)^i \qquad (1)$$

We must also have:

$$\sum_{i=0}^{\infty} \pi_i = 1 \Rightarrow \pi_0 \sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i = 1 \quad (2)$$

But
$$\sum_{i=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^i < \infty \Leftrightarrow \frac{\lambda}{\mu} < 1 \Leftrightarrow \lambda < \mu.$$

(ii) It suffices to show that the uniformized CTMC is positive recurrent.

From Eq. (2) in page 104 of the Primer, we have:

$\forall i \in \{0, 1, 2, \ldots\}$,

$$\pi_i \, E[T_{ii}] = \sum_{\ell=0}^{\infty} \pi_\ell \tau_\ell \Rightarrow$$

$$\Rightarrow \pi_i \, E[T_{ii}] = \frac{1}{\lambda+\mu} \sum_{\ell=0}^{\infty} \pi_\ell \Rightarrow$$

$$\Rightarrow E[T_{ii}] = \frac{1}{\lambda+\mu} \frac{1}{\pi_i} < \infty$$

Remark: An intuitive interpretation of the characterization of $E[T_{ii}]$ in the previous page is as follows:

The irreducibility and the positive recurrence of the embedded DTMC of the uniformized process implies that the length of the recurrence cycles w.r.t. state x_i for this process is equal to $1/\pi_i$.

Furthermore, the uniformized nature of the considered CTMC implies that the average sojourn time at every state visited during this recurrence cycle is $1/\nu = 1/(\lambda+\mu)$.

Hence, the result.

Remark: From the uniformization theory presented in class, we also know that if $\lambda<\mu$, the stationary distribution π of the embedded DTMC of the uniformized CTMC is also the limiting distribution p of the original CTMC.

From Eqs (1) and (2), π can be computed as follows:

Set $\quad \rho \equiv \lambda/\mu < 1 \quad (3)$

Then, from (2)

$$\pi_0 \sum_{i=0}^{\infty} \rho^i = 1 \iff \pi_0 \frac{1}{1-\rho} = 1 \iff$$

$$\iff \pi_0 = 1-\rho \quad (4)$$

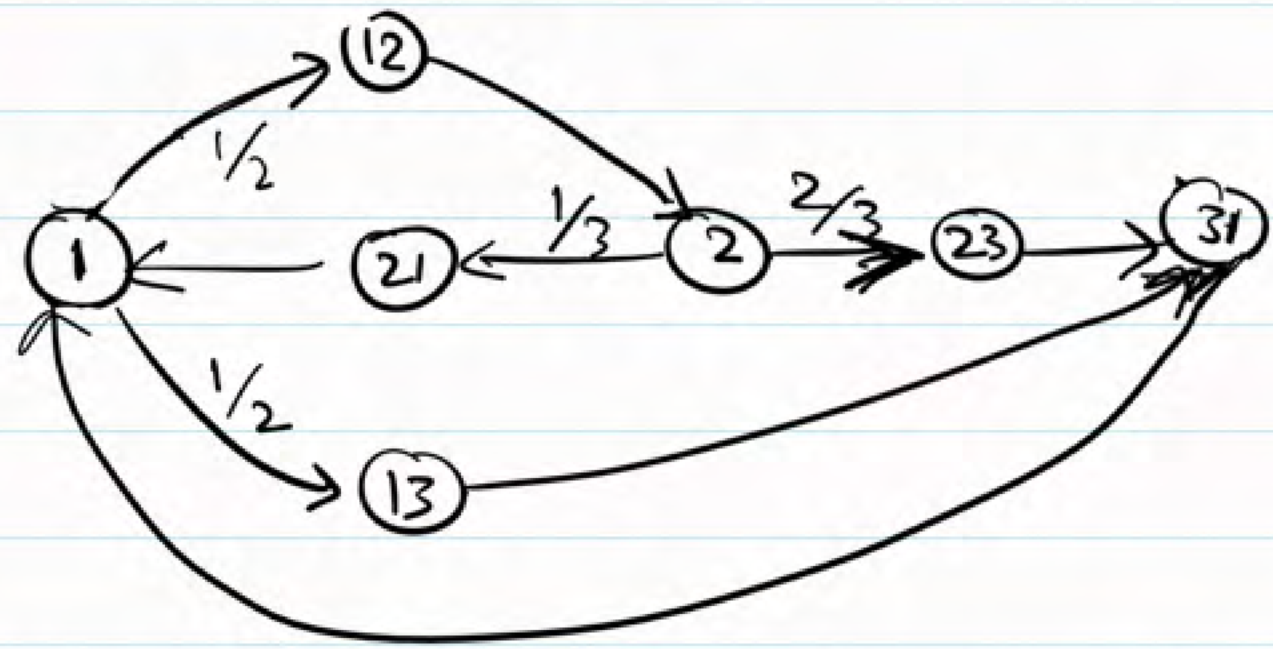
and from (1), (3) and (4)

$$\forall i \in \{0, 1, 2, \dots\}, \quad \pi_i = (1-\rho)\rho^i \quad (5)$$

Hence, π is a geometric distribution with parameter $1-\rho$.

Problem 5:

The one-step transitional dynamics of the shuttle among the three locations can be modeled as follows:



The corresponding one-step transition prob. matrix $\hat{P}$ is

$$
\hat{P} = \begin{array}{c} \\ 1 \\ 12 \\ 13 \\ 2 \\ 21 \\ 23 \\ 31 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 12 & 13 & 2 & 21 & 23 & 31 \end{array} \\
\left[\begin{array}{ccccccc}
 & 1/2 & 1/2 & & & & \\
 & & & 1 & & & \\
 & & & & & & 1 \\
 & & & & 1/3 & 2/3 & \\
1 & & & & & & \\
 & & & & & & 1 \\
1 & & & & & &
\end{array}\right]
\end{array}
$$

Clearly, the considered DTMC is irreducible, and since it is finite-state, it is also positive recurrent. Hence, it has a stationary distr. π that must satisfy the following equations:

$$\begin{cases} \pi_1 = \pi_{21} + \pi_{31} \\ \pi_{12} = \frac{1}{2}\pi_1 \\ \pi_{13} = \frac{1}{2}\pi_1 \\ \pi_2 = \pi_{12} \\ \pi_{21} = \frac{1}{3}\pi_2 \\ \pi_{23} = \frac{2}{3}\pi_2 \end{cases} \Rightarrow \begin{cases} \pi_{12} = \frac{1}{2}\pi_1 \\ \pi_{13} = \frac{1}{2}\pi_1 \\ \pi_2 = \frac{1}{2}\pi_1 \\ \pi_{21} = \frac{1}{6}\pi_1 \\ \pi_{23} = \frac{2}{6}\pi_1 \\ \pi_{31} = \frac{5}{6}\pi_1 \end{cases}$$

Also, $\sum_i \pi_i = 1 \Rightarrow$

$$\Rightarrow \pi_1 \left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{6} + \frac{2}{6} + \frac{5}{6}\right) = 1$$

$$\Rightarrow \pi_1 = 6/23$$

and

$\pi_{12} = \pi_{13} = \pi_2 = 3/23$; $\pi_{21} = 1/23\, \pi_1$; $\pi_{23} = 2/23\, \pi_1$;

$\pi_{31} = 5/23\, \pi_1$

The expected sojourn times of the various states are:

$$\tau_1 = \tau_2 = 0 \text{ min}$$

$$\tau_{12} = \tau_{21} = 20 \text{ min}$$

$$\tau_{13} = \tau_{31} = 30 \text{ min}$$

$$\tau_{23} = 30 \text{ min}$$

The continuous-time dynamics of the shuttle among the three locations constitute a semi-Markov process with its limiting distribution p defined from π_i and τ_i according to Eq. (3) in page 105 of the Primer.

First we have:

$$\sum_i \pi_i \tau_i =$$

$$= \frac{6}{23} \cdot 0 + \frac{3}{23} \cdot 20 + \frac{3}{23} \cdot 30 +$$

$$\frac{3}{23} \cdot 0 + \frac{1}{23} \cdot 20 + \frac{2}{23} \cdot 30 +$$

$$+ \frac{5}{23} \cdot 30 = \frac{380}{23}$$

Hence,

$$P_1 = P_2 = 0$$

$$P_{12} = \frac{\frac{3}{23} 20}{380/23} = 6/38$$

$$P_{13} = \frac{\frac{3}{23} 30}{380/23} = 9/38$$

$$P_{21} = \frac{\frac{1}{23} \cdot 20}{380/23} = 2/38$$

$$P_{23} = \frac{\frac{2}{23} \cdot 30}{380/23} = 6/38$$

$$P_{31} = \frac{\frac{5}{23} \cdot 30}{380/23} = 15/38$$

Now we are ready to answer the questions of the problem.

(i) The corresponding probabilities are:

1: $P_{12} + P_{13} = 6/38 + 9/38 = 15/38$

2: $P_{21} + P_{23} = 2/38 + 6/38 = 8/38 = 4/19$

3: $P_{31} = 15/38$

(ii) This probability is $P_{12} = 6/38 = 3/19$

(iii) This fraction of time is $P_{23} = 6/38 = 3/19$