ISYE 7201: Production & Service Systems
Spring 2023
Instructor: Spyros Reveliotis
1st Midterm Exam (Take Home)
Release Date: January 27, 2023
Due Date: January 31, 2023

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

Please, send me your responses as a pdf file attached to an email. Name the pdf file by your last name (only). The pdf file can be a scan or photos of a hand-written document, but, please, write your answers clearly and thoroughly. Also, make sure that the pdf file is not too big; you can reduce the size of your file by loading it into Adobe Acrobat and saving it with the "reduced" size option before emailing it to me.

Finally, report any external sources (other than your textbook) that you referred to while preparing the solutions.

Problem 1 (20 pts): Let $U = (U_n,\ n \in \mathbb{Z})$ consist of independent random variables, each uniformly distributed on the interval $[0, 1]$, and let $X = (X_k,\ k \in \mathbb{Z})$ be defined by $X_k = max\{U_{k-1}, U_k\}$.

i. (5 pts) Sketch a typical sample path of the process X.

ii. (5 pts) Is X a stationary random process?

iii. (10 pts) Is X a discrete-time Markov process?

Problem 2 (20 points): Consider a hospital where the rooms are organized in three different types: general care, special care, and intensive care. Based on past data, 60% of arriving patients are initially admitted into the general care category, 30% in the special care category, and 10% in the intensive care. A "general care" patient has a 55% chance of being released healthy the following day, a 30% chance of remaining in the general care, and a 15% of being moved to the special care. A "special care" patient has a 10% chance of being released the following day, a 20% chance of being moved to the general care, a 10% chance of being moved to the intensive care, and a 5% chance of dying during the day. An "intensive care" patient is never released from the hospital directly from the intensive care unit (ICU), but is always moved to another facility first. The probabilities for being moved to general care, special care, or remaining in the intensive care are, respectively, 5%, 30% and 55%.

Let $\{X_k, k \geq 0\}$ be the stochastic process that traces the daily placement of an admitted patient with respect to the aforementioned three categories.

i. (5 pts) Argue that this stochastic process is a DTMC, and provide the one-step transition probability matrix P.

ii. (5 pts) What is the probability that an arriving patient will be released on the fifth day of his admission?

iii. (5 pts) Compute the expected sojourn time (in number of days) for any single visit to the ICU by any given patient.

iv. (5 pts) What is the average number of patients in the ICU if, during a typical day, 100 patients are admitted to the hospital?

Problem 3 (20 points): A manufacturing workstation consisting of a single server operates according to an 8-hour daily shift. At the end of each shift, the server is tested, and if everything is found satisfactory, then the server will continue its operation in the following day. But there is a probability $p_1 = 0.2$ that the performed test will indicate some degradation in the operational condition of the server. For such a case, the management of the considered workstation currently considers two possibilities:

1. Arrange for some preventive maintenance for the server to take place during the following day, and have the server resume its operation the day after.

2. Continue running the server at its degraded state, monitoring at the end of every shift for any further degradation. If such a degradation is diagnosed, the server is sent for maintenance in the next day, and it resumes its operation the day after the next day fully recovered. Historical data indicates that the server probability of failing the test while operating in its first degraded mode is $p_2 = 0.5$.

Also, the management estimates that when the server is running in its healthy condition, it generates a revenue of 10K, while operation in the degraded mode generates a revenue of 5K. Furthermore, conducting a session of preventive maintenance on the server costs 3K.

You need to help the workstation management choose between the two strategies of preventive maintenance for the server that are defined above.

Problem 4 (20 points): You enter a game of chance with a bank of x dollars. The game is played in rounds and, at every round, you bet one dollar, which you can double with probability 0.5 and you can lose with the remaining probability. Your intention is to double your original bank of x dollar before you leave the game. You will also have to leave the game if you get broke. Show that this game can be modeled as a DTMC and compute the probability that you will meet your objective.

Hint: There is an easy solution for this problem.

Problem 5 (20 points): Consider a stochastic matrix P with all its columns adding up to 1.0. Such a matrix is called *doubly stochastic*.

i. (10 pts) Show that a finite-state irreducible Markov chain with a doubly stochastic one-step probability matrix has a uniform stationary distribution.

ii. (10 pts) Consider an iterative scoring process involving n experts who try to reach to a consensus. At each iteration, the i-th expert communicates her scores to a subset of experts $\mathcal{E}(i) \subset \{1, 2, \ldots, n\} \setminus \{i\}$ and she also receives the scores of these experts. Furthermore, the simple undirected graph G with node set $V = \{1, \ldots, n\}$ and edge set $E = \{\{i, j\} : j \in \mathcal{E}(i) \wedge i \in \mathcal{E}(j)\}$ is connected.

   At the first iteration, each expert $i \in \{1, \ldots, n\}$ comes up with a score $s_i(1)$ and broadcasts this score to every expert in $\mathcal{E}(i)$. At each subsequent iteration $t \geq 2$, expert i revises her own score to the average of her score and the scores that she received from her communicating experts during this iteration according to the following formula:

$$s_i(t) = \left(1 - \frac{d(i)}{K}\right) s_i(t-1) + \frac{1}{K} \sum_{j: i \in \mathcal{E}(j)} s_j(t-1)$$

   where $d(i)$ is the degree of node i in G and $K > \max_i\{d(i)\}$. The revised scores are broadcasted again, and the process proceeds to the next iteration.

Use the result of part (i) to show that this scoring process converges to the average of the initial expert scores, i.e.,

$$\forall i \{1, \ldots, n\}, \quad \lim_{t \to \infty} s_i(t) = \frac{1}{n} \sum_{j=1}^{n} s_j(1)$$

It is also interesting to notice that the above result is independent of the exact value of the parameter K (as long as $K > \max_i\{d(i)\}$).

Finally, please, explain clearly all your answers.

Problem 1

(i) Suppose that a realization of the iid sequence $\{U_k, k \geq 0\}$ is

$$\langle 0.5, 0.4, 0.3, 0.6, 0.8, 0.3, 0.7, 0.9, \ldots \rangle$$

Then

$$x_1 = \max\{0.5, 0.4\} = 0.5$$
$$x_2 = \max\{0.4, 0.3\} = 0.4$$
$$x_3 = \max\{0.3, 0.6\} = 0.6$$
$$x_4 = \max\{0.6, 0.8\} = 0.8$$
$$x_5 = \max\{0.8, 0.3\} = 0.8$$
$$x_7 = \max\{0.3, 0.7\} = 0.7$$
$$x_8 = \max\{0.7, 0.9\} = 0.9$$

Notice that for pairs (U_{k-1}, U_k) where the sequence $\{U_k\}$ is decreasing, $x_k = U_{k-1}$ and for such pairs where the sequence $\{U_k\}$ is increasing, $x_k = U_k$. A plot of the above sample path is as follows:

Notice that $\{X_k\}$ presents less fluctuating than $\{V_k\}$.

(ii) For stationarity, we need

$$\forall k, \quad P(X_{k+i_1} \le x_1, \ X_{k+i_2} \le x_2, \ldots, X_{k+i_n} \le x_n)$$
$$= P(X_{i_1} \le x_1, \ X_{i_2} \le x_2, \ldots, X_{i_n} \le x_n)$$

This equation holds since $\{V_k\}$ is a sequence of iid r.v.'s.

More formally:

$$\forall k, \quad P(X_{k+i_1} \le x_1, X_{k+i_2} \le x_2, \dots, X_{k+i_n} \le x_n)$$

$$= P(U_{k+i_1-1} \le x_1, U_{k+i_1} \le x_1,$$
$$U_{k+i_2-1} \le x_2, U_{k+i_2} \le x_2, \dots,$$
$$U_{k+i_n-1} \le x_n, U_{k+i_n} \le x_n) \overset{(*)}{=}$$

$$\overset{(*)}{=} P(U_{i_1-1} \le x_1, U_{i_1} \le x_1,$$
$$U_{i_2-1} \le x_2, U_{i_2} \le x_2, \dots,$$
$$U_{i_n-1} \le x_n, U_{i_n} \le x_n) =$$

$$= P(X_{i_1} \le x_1, X_{i_2} \le x_2, \dots, X_{i_n} \le x_n)$$

In the above derivation $(*)$ holds because the sequence $\{U_k\}$ is iid.

(iii) We show that $\{X_n\}$ is not a DT-MC by showing that

$$P(X_3 \leq 0.5 \mid X_2 = 0.8 ; X_1 = 0.8) \neq$$

$$P(X_3 \leq 0.5 \mid X_2 = 0.8 ; X_1 = 0.7)$$

Indeed,

$$P(X_3 \leq 0.5 \mid X_2 = 0.8 ; X_1 = 0.7) = \emptyset$$

since $X_2 > X_1$ implies that $U_2 = 0.8$

and $X_3 = \max\{U_2, U_3\} \geq U_2$

On the other hand, $X_2 = 0.8 \wedge X_1 = 0.8$ allows U_2 to have any value less than or equal to 0.8.

Hence, $P(X_3 \leq 0.5 \mid X_2 = 0.8 ; X_1 = 0.8) =$

$$= P(\max\{U_3, U_2\} \leq 0.5 \mid U_2 \leq 0.8) =$$

$$= P(U_3 \leq 0.5) \cdot P(U_2 \leq 0.5 \mid U_2 \leq 0.8) > 0.$$

Problem 2

(i) From the problem description it is clear that the possible daily states of any patient admitted to the hospital after his admission are

- G: general care
- S: special care
- I: intensive care
- R: released
- F: fatal

Also, states R and F are terminal (or absorbing) states. The one-step trans. prob. matrix P defining the process transitions among these states is:

$$P = \begin{array}{c} \\ G \\ S \\ I \\ R \\ F \end{array} \begin{array}{c} \begin{array}{ccccc} G & S & I & R & F \end{array} \\ \begin{bmatrix} 0.3 & 0.15 & 0 & 0.55 & 0 \\ 0.2 & 0.1 & 0.1 & 0.1 & 0.5 \\ 0.05 & 0.3 & 0.55 & 0 & 0.1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad (1)$$

The initial-state prob. distribution $\pi(0)$ is

$$\pi(0) = (\pi_G \; \pi_S \; \pi_I \; \pi_R \; \pi_F) = (0.6 \; 0.3 \; 0.1 \; 0 \; 0) \qquad (2)$$

(ii) Let q denote the considered probability. Then, using the Primer notation, q can be expressed as

$$q = \pi_G(4) \cdot P_{GR} + \pi_S(4) P_{SR}$$

$$\overset{(1)}{=} 0.55\, \pi_G(4) + 0.1\, \pi_S(4) \qquad (3)$$

Also,

$$\pi(4) = \pi(0) P^4 =$$

$$= (0.6 \;\; 0.3 \;\; 0.1 \;\; 0 \;\; 0) \begin{bmatrix} 0.3 & 0.15 & 0 & 0.55 & 0 \\ 0.2 & 0.1 & 0.1 & 0.1 & 0.5 \\ 0.05 & 0.3 & 0.55 & 0 & 0.1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^4 =$$

$$= (0.0258 \;\; 0.0244 \;\; 0.0272 \;\; 0.609 \;\; 0.3137)$$

Hence,

$$q = 0.55 \cdot 0.0258 + 0.1 \cdot 0.0244 = 0.01663$$

(iii) The probability that a patient in ICU will leave this unit the next day is $1-0.55 = 0.45$

We can think of this probability as the "success probability" p_s of a Bernoulli trial with success implying the patient exit from the ICU..

Since the expected number of Bernoulli trials till success is equal to $1/p_s$, the expected number of days that the patient will stay in ICU when he enters this unit is $1/0.45 = 2.22$

(iv) Since admitted patients move independently during their sojourn in the hospital, we can get the requested number by

$$\sum_{k=0}^{\infty} 100\, \pi_I(k) = 100 \sum_{k=0}^{\infty} \pi_I(k)$$

In the above expression, the term $100\, \pi_I(k)$ is the expected number of patients in ICU among the 100 patients admitted k days in the past

We also have:

$$\pi_I(k) = \pi(0)\cdot P^k[\cdot, I]$$

Next we tabulate this computation for $k = 0, 1, \ldots, 13$.

k	$\left(P^k[\cdot, I]\right)^T$	$\pi_I(k)$
0	$[0\ \ 0\ \ 1\ \ 0\ \ 0]$	0.1
1	$[0\ \ 0.1\ \ 0.55\ \ 0\ \ 0]$	0.085
2	$[0.015\ \ 0.065\ \ 0.3325\ \ 0\ \ 0]$	0.0618
3	$[0.0143\ \ 0.0428\ \ 0.2031\ \ 0\ \ 0]$	0.0417
4	$[0.0107\ \ 0.0274\ \ 0.1253\ \ 0\ \ 0]$	0.0272
5	$[0.0073\ \ 0.0174\ \ 0.0777\ \ 0\ \ 0]$	0.0174
6	$[0.0048\ \ 0.011\ \ 0.0483\ \ 0\ \ 0]$	0.0110
7	$[0.0031\ \ 0.0069\ \ 0.0301\ \ 0\ \ 0]$	0.0069
8	$[0.002\ \ 0.0043\ \ 0.0187\ \ 0\ \ 0]$	0.0043
9	$[0.0012\ \ 0.0027\ \ 0.0117\ \ 0\ \ 0]$	0.0027
10	$[0.0008\ \ 0.0017\ \ 0.0073\ \ 0\ \ 0]$	0.0017
11	$[0.0005\ \ 0.0011\ \ 0.0046\ \ 0\ \ 0]$	0.0011
12	$[0.0003\ \ 0.0007\ \ 0.0029\ \ 0\ \ 0]$	6.66×10^{-4}
13	$[0.0002\ \ 0.0004\ \ 0.0018\ \ 0\ \ 0]$	4.16×10^{-4} +
		0.3619

Hence, $100\sum_{k=0}^{\infty}\pi_I(k) \simeq 100\sum_{k=0}^{13}\pi_I(k) = 36.19$

There is an alternative and, in fact, more accurate approach to compute the expected number of patients at ICU.

Define X_i, $i \in \{G, S, I\}$ as the expected number of patients in the corresponding facility.

Then, assuming 100 admissions per day, these three quantities must satisfy the following "balance equations":

$$X_G = 0.6 \cdot 100 + 0.3 X_G + 0.2 X_S + 0.05 X_I \quad (4)$$
$$X_S = 0.3 \cdot 100 + 0.15 X_G + 0.1 X_S + 0.3 X_I \quad (5)$$
$$X_I = 0.1 \cdot 100 + 0 X_G + 0.1 X_S + 0.55 X_I \quad (6)$$

Then, from (6),

$$X_S = 4.5 X_I - 100 \quad (7)$$

and from (4) and (7),

$$X_G = 1.36 X_I + 57.14 \quad (8)$$

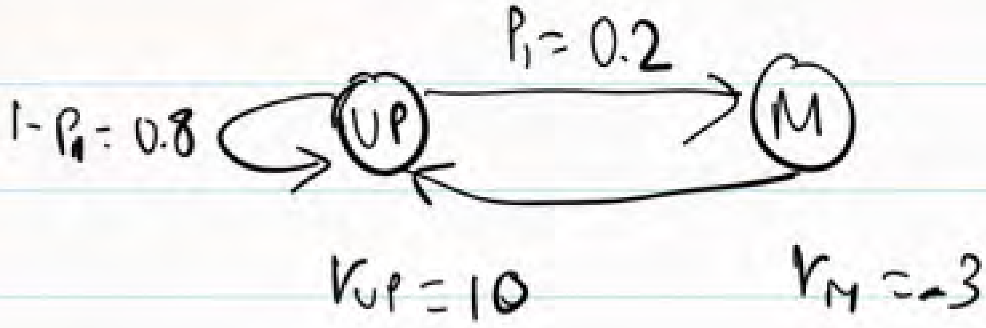
Finally, from (5), (7) and (8)

$$X_I = 36.22$$

So, our previous approximation was pretty good.

Problem 3

The first of the two strategies described in this problem can be modeled by the following DT-MC:



$$P = \begin{array}{c} UP \\ M \end{array} \begin{bmatrix} 0.8 & 0.2 \\ 1 & 0 \end{bmatrix}$$

In the above STD, state UP models the running state of the considered workstation and state M is the state where it is in maintenance. Also r_{UP} and r_M model the revenue generated when the workstation spends a single period at the corresponding state.

The above MC is irreducible and finite-state, and therefore, positive recurrent. Hence, it has a stationary distribution π, which can be computed from the following equations:

$$\begin{cases} (\pi_{UP} \ \pi_M) = (\pi_{UP} \ \pi_M) P \\ \pi_{UP} + \pi_M = 1 \end{cases} \Rightarrow$$

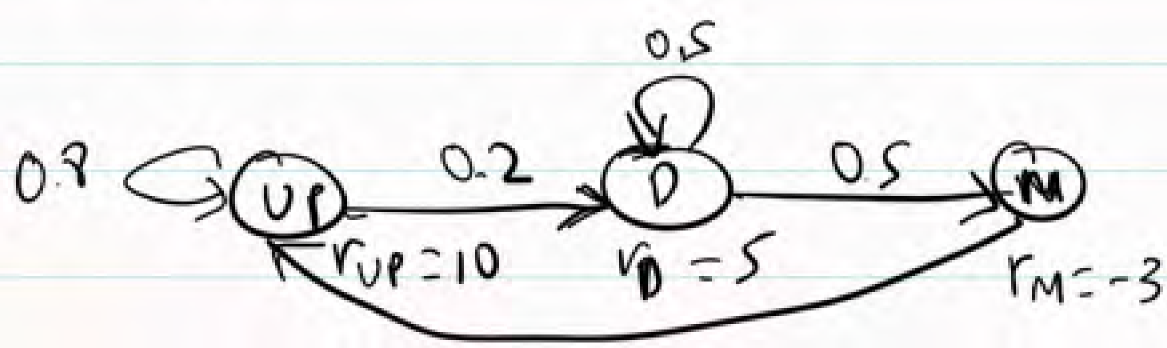
$$\begin{cases} \pi_{UP} = 0.8\,\pi_{UP} + \pi_M \\ \pi_{UP} + \pi_M = 1 \end{cases} \Rightarrow \begin{cases} \pi_M = 0.2\,\pi_{UP} \\ 1.2\,\pi_{UP} = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \pi_{UP} = 0.8\overline{3} \simeq 0.833 \\ \pi_M = 0.1\overline{6} \simeq 0.1667 \end{cases}$$

Then, the long-run daily revenue resulting from this strategy is:

$$DR_1 = \pi_{UP}\, r_{UP} + \pi_M\, r_M =$$
$$= 0.833 \cdot 10 + 0.1667 \cdot (-3) = 7.833$$

The second strategy is modeled by the following DT-MC:



$$P = \begin{array}{c} UP \\ D \\ M \end{array} \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}$$

where state D models the degraded operational mode.

The stationary distribution for this DT-MC is computed by:

$$\begin{cases} \pi_{UP} = 0.8\,\pi_{UP} + \pi_M \\ \pi_M = 0.5\,\pi_D \\ \pi_{UP} + \pi_D + \pi_M = 1 \end{cases} \Rightarrow \begin{cases} \pi_{UP} = 0.625 \\ \pi_D = 0.25 \\ \pi_M = 0.125 \end{cases}$$

The corresponding daily revenue is:

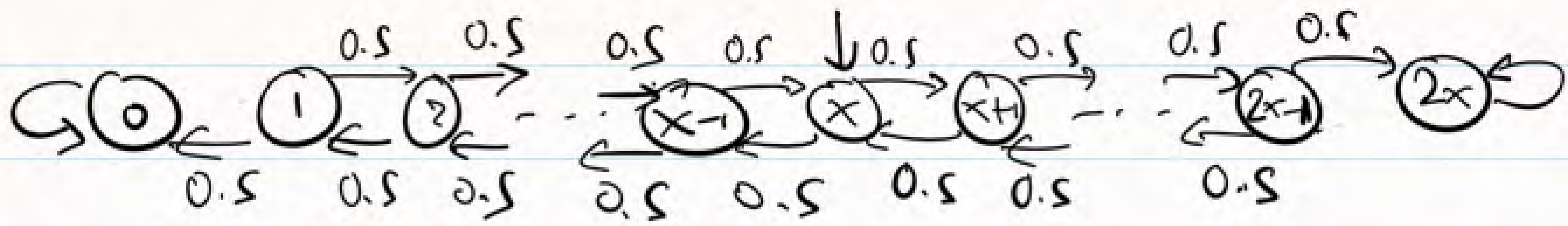
$$DR_2 = \pi_{UP}\, r_{UP} + \pi_D\, r_D + \pi_M\, r_M =$$

$$= 0.625 \cdot 10 + 0.25 \cdot 5 + 0.125\,(-3) = 7.125$$

Hence, Strategy 1 is more ludicrous.

Problem 4

The game dynamics can be modelled by the following DT-MC:



In this MC the initial state is state x, and the states 0 and $2x$ are absorbing states.

Furthermore, it is clear from the above STD that the process behavior is completely symmetrical with respect to its absorbing states.

Therefore, the absorption probability for each of these two states is 0.5.

Problem 5

(i) Since the considered DTMC is irreducible and finite state, it is also positive recurrent, and therefore, it has a stationary distribution π. Furthermore, π is unique. Let n denote the number of states of this MC. Then, it suffices to show that $\pi = (1/n, \ldots, 1/n)$ satisfies the equation $\pi = \pi \cdot P$.

Indeed for any $i \in \{1, \ldots, n\}$, we have:

$$\pi_i = \sum_{j=1}^{n} \pi_j P_{ji} = \sum_{j=1}^{n} \frac{1}{n} P_{ji} = \frac{1}{n} \sum_{j=1}^{n} P_{ji} \overset{*}{=}$$

$$\overset{*}{=} \frac{1}{n} (1) = \frac{1}{n}$$

Where $*$ holds because P is doubly stochastic.

(ii) Let $\underline{s}(t) = (s_1(t), \dots, s_n(t))^T$.

Then, these vectors are obtained from the following recursion:

$$\underline{s}(t) = P \cdot \underline{s}(t-1) \qquad (1)$$

where P is an $n \times n$ matrix with

$\forall i$, $\quad p_{ii} = 1 - \frac{d(i)}{K} \in (0,1)$

$$\forall i,j,\ i \neq j, \quad p_{ij} = \begin{cases} 1/K, & \text{if } i \text{ and } j \text{ communicate} \\ 0, & \text{otherwise} \end{cases}$$

For the row sums of matrix P we have:

$$\sum_{j=1}^{n} p_{ij} = p_{ii} + \sum_{j \neq i} p_{ij} \overset{(*)}{=} 1 - \frac{d(i)}{K} + d(i)\frac{1}{K} = 1$$

where (*) results from the fact that $d(i)$ is the degree of node i in G, and therefore, the number of experts communicating with expert i.

Hence, P is stochastic.

Also, the column sums of P are

$$\sum_{i=1}^{n} P_{ij} \overset{*}{=} \sum_{i=1}^{n} P_{ji} \overset{**}{=} 1$$

In the above equation,

$*$ results from the fact that P is symmetric, and

$**$ results from the previous result that P is stochastic.

Hence, P is doubly stochastic.

Since the considered graph G is connected, the DT-MC defined by P is irreducible. Since it is also finite-state, it is positive recurrent. The positive diagonal elements of P also imply that this MC has a limiting distribution π and

$$\lim_{t \to \infty} P^t = \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix} \quad (2)$$

The doubly stochastic property of P also implies that π is uniform (from part (i) of this problem).

Finally from (1) and (2),

$$\forall i, \quad \lim_{t \to \infty} s_i(t) = \pi \cdot \underline{s}(1) = \frac{1}{n} \sum_{j=1}^{n} s_j(1)$$