ISYE 7201: Production & Service Systems
Spring 2023
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Final Exam (Take Home)
Release Date: April 27, 2023
Due Date: April 30, 2023

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

Please, send me your responses as a pdf file attached to an email. Name the pdf file by your last name (only). The pdf file can be a scan or photos of a hand-written document, but, please, write your answers clearly and thoroughly. Also, make sure that the pdf file is not too big; you can reduce the size of your file by loading it into Adobe Acrobat and saving it with the "reduced" size option before emailing it to me.

Finally, report any external sources (other than your textbook) that you referred to while preparing the solutions.

Problem 1 (25 points): Consider a serial workflow consisting of m single-server workstations, WS_i, $i \in \{1, \dots, m\}$, with corresponding processing times exponentially distributed with rate μ_i. This workflow is fed by a Poisson process with rate $\lambda < \min_i \mu_i$.

i. (5 pts) Explain that the operation of this workflow is stable.

ii. (10 pts) Assuming that this workflow is operating in its limiting regime, determine the probability that there are n items in it.

iii. (5 pts) Specialize the result in part (ii) assuming that $\mu_i = \mu$, $\forall i$.

iv. (5 pts) Compute the probability required in part (ii) when $m = 3$, $n = 10$, $\lambda = 5\ hr^{-1}$, $\mu_1 = \mu_3 = 6\ hr^{-1}$ and $\mu_2 = 7\ hr^{-1}$.

Problem 2 (25 points): Consider a counter with a single clerk that serves customers arriving according to a Poisson process with rate $\lambda_c = 8\ hr^{-1}$ following a FCFS protocol. The serving of any given customer by the clerk involves the processing of some paperwork and the required time for this paperwork is normally distributed with mean $t_c = 5\ min$ and st. deviation $\sigma_c = 3\ min$. However, while serving a customer, the clerk might need to make some clarifying phone calls, and past observations have shown that these phone calls occur according to a Poisson process with rate $\lambda_p = 5\ hr^{-1}$ and their duration is exponentially distributed with mean $t_p = 3\ min$.

i. (5 pts) Show that the counter will serve the arriving customers in a stable manner.

ii. (5 pts) What is the throughput with which customers are serviced at this counter?

iii. (15 pts) Also, provide a Mean Value Analysis (MVA) of this counter with respect to the serviced customers; in particular, compute: (i) $E[W]$, the expected waiting time by any customer before she is picked up for service by the clerk; (ii) $E[S]$, the expected total time spent by a customer at this counter; (iii) $E[X_q]$, the expected number of customers waiting for service; and (iv) $E[X]$, the expected number of customers present at the counter when considering also the potential customer in service.

Problem 3 (25 points): Consider a counter with a single clerk that serves customers arriving according to a Poisson process with rate $\lambda_c = 8\ hr^{-1}$ following a FCFS protocol. Besides serving the arriving customers, the clerk also answers incoming phone calls that occur according to a Poisson process with rate $\lambda_p = 5\ hr^{-1}$. Furthermore, the phone calls have *preemptive priority* over the served customers, i.e., whenever the phone rings while a customer is in service, the clerk must answer the phone call and subsequently she resumes servicing the customer. Customer service times are normally distributed with mean $t_c = 5\ min$ and st. deviation $\sigma_c = 3\ min$, while the duration of each phone call is exponentially distributed with mean $t_p = 3\ min$.

i. (5 pts) Show that the counter will serve the arriving customers in a stable manner.

ii. (5 pts) What is the throughput with which customers are serviced at this counter?

iii. (15 pts) Also, provide a Mean Value Analysis (MVA) of this counter with respect to the serviced customers; in particular, compute: (i) $E[W]$, the expected waiting time by any customer before she is picked up for service by the clerk; (ii) $E[S]$, the expected total time spent by a customer at this counter; (iii) $E[X_q]$, the expected number of customers waiting for service; and (iv) $E[X]$, the expected number of customers present at the counter when considering also the potential customer in service.

Problem 4 (25 points): Consider a local shop operating on a 12-hour daily shift that serves orders arriving according to a Poisson process with rate $\lambda = 5$ per day. The shop processes the incoming orders according to a FCFS protocol, one order at a time, and each order is filled by manufacturing and delivering a certain part. The manufacturing of a part takes two hours, but a part can be damaged while in processing, in which case, a new part must be started for the satisfaction of the corresponding order.

Assuming that such a catastrophic failure can occur uniformly over the 2-hour interval that is required for the complete processing of the part, do the following:

i. (10 pts) Compute the maximum failing probability $\bar{p}$ for the processing of any single part that will lead to a stable operation of the considered shop.

ii. (15 pts) Perform a mean-value analysis of the shop operation assuming that the failing probability is $p = 0.5\bar{p}$. In particular, provide the utilization, the throughput, the expected lead time for an order, and the average number of standing orders at any time point.

Problem 1:

(i) For workstation WS_1 we have:

$$\rho_1 = \lambda/\mu_1 < \lambda/\min_i \mu_i < 1$$

by the working assumption.

Next suppose that the first k workstations of the line are stable. Hence, from the material-conservation law, the arrival rate at workstation WS_{k+1} is λ. Also,

$$\rho_{k+1} = \lambda/\mu_{k+1} < \lambda/\min_i \mu_i < 1$$

and therefore, the stability of the entire process has been established by induction.

(ii) The considered workflow is a Jackson network which, according to part (i), is stable. From the theorem characterizing the limiting distribution of a stable Jackson network, every workstation WS_i operates in this regime as a stable M/M/1 queue with arrival rate λ and proc. rate μ_i. Hence, the requested probability is

$P = \Pr[n \text{ items in the line}] =$

$$= \sum_{\substack{q=(q_1,\dots,q_m):\\ \sum_{i=1}^m q_i = n}} \prod_{i=1}^m (1-\rho_i)\rho_i^{q_i}$$

$$= \left[\prod_{i=1}^m (1-\rho_i)\right]\left[\sum_{\substack{q=(q_1,\dots,q_m):\\ \sum_{i=1}^m q_i = n}} \prod_{i=1}^m \rho_i^{q_i}\right] \qquad (1)$$

(iii) When $\mu_i = \mu, \forall i$, we also have $\rho_i = \lambda/\mu = \rho, \forall i$.
Hence, Eq. (1) becomes

$P = \Pr[n \text{ items in the line}] =$

$$= (1-\rho)^m \sum_{\substack{q=(q_1,\dots,q_m):\\ \sum_{i=1}^m q_i = n}} \rho^n =$$

$$= (1-\rho)^m \rho^n \left[\sum_{q=(q_1,\dots,q_m):\ \sum_{i=1}^n q_i = n} 1\right] = (1-\rho)^m \rho^n \binom{n+m-1}{m-1} \qquad (2)$$

The last step of Eq. (2) recognizes that the involved summation essentially counts all the possible ways for splitting the n customers to the m workstations, and applies the corresponding result in page 208 of the Primer.

(iv) In this case we have:

$$\rho_1 = \lambda/\mu_1 = \lambda/\mu_3 = \rho_3 = \frac{5}{6} \simeq 0.833$$

$$\rho_2 = \lambda/\mu_2 = 5/7 \simeq 0.714$$

Hence, $\prod_{i=1}^{3}(1-\rho_i) = (1-0.833)^2(1-0.714) =$

$$\simeq 0.008 \qquad (3)$$

For the second factor in Eq. (1) we can use Buzen's convolution algorithm, in particular the special case that is presented in page 213 of the Primer.

In this case,

$$g_j(l) = \rho_j^l \qquad \begin{cases} j = 1,2,3 \\ l = 0,1,\ldots,10 \end{cases} \qquad (4)$$

and the recursion of page 213 in the Primer becomes:

$$\mathcal{G}(j,q) = p_j\, \mathcal{G}(j,q-1) + \mathcal{G}(j-1,q) \quad (5)$$

with boundary conditions:

$$\mathcal{G}(j,\emptyset) = 1, \forall j \quad \wedge \quad \mathcal{G}(1,q) = p_1^q, \forall q \quad (6)$$

We need to compute $\mathcal{G}(3,10)$.

This computation is tabulated as follows:

$j \backslash q$	0	1	2	3	4	5	6	7	8	9	10
1	1	p_1	p_1^2	p_1^3	p_1^4	p_1^5	p_1^6	p_1^7	p_1^8	p_1^9	p_1^{10}
2	1	$\mathcal{G}(2,1)$	-		-	—	—	-			$\mathcal{G}(2,10)$
3	1	$\mathcal{G}(3,1)$	-		—		—	-			$\mathcal{G}(3,10)$

We can execute the above computation, using Eq. (5), easily in Excel. A printout of the obtained spreadsheet is appended at the end of this document.

From the spreadsheet we can see that

$$G(3,10) \simeq 6.87 \qquad (7)$$

Finally, from Equations (3) and (7), we get:

$$p = 0.008 \cdot 6.87 \simeq 0.055$$

Problem 2:

This problem can be referred to the model of the M/G/1 queue with preemptive non-destructive outages discussed in pages 259 - 266 of the Primer. In this case, the experienced disruptions are the clarifying calls that must be made during the processing of the paperwork of a customer. By applying this theory, we have the following:

(i) Availability $A = \dfrac{1/\lambda_p}{1/\lambda_p + t_p} = \dfrac{60/5}{60/5+3} = \dfrac{12}{15} = 0.8$

Then, $\rho = \lambda_c \dfrac{t_c}{A} = 8 \cdot \dfrac{5/60}{0.8} = \dfrac{5}{6} = 0.833 < 1$

Hence, the considered operation is stable.

(ii) Since this operation is stable,

$TH_c = \lambda_c = 8 \text{ hr}^{-1}$.

(iii) First we must compute t_{ce} and c_{ce}^2.

Applying the corresponding formulae derived in the Primer, we get:

$$t_{ce} = \frac{t_c}{A} = \frac{5\,\text{min}}{0.8} = 6.25\,\text{min}$$

$$c_{ce}^2 = c_c^2 + (1+c_p^2)\,A(1-A)\,\frac{t_p}{t_c} =$$

$$= \left(\frac{3}{5}\right)^2 + (1+1)\,0.8\,(1-0.8)\,\frac{3}{5} = 0.552.$$

Then, from the Pollacek-Kintchine formulae for the M/G/1 queue, we get:

$$E[W] = \frac{1+c_{ce}^2}{2}\,\frac{\rho}{1-\rho}\,t_{ce} =$$

$$= \frac{1+0.552}{2}\,\frac{0.833}{1-0.833}\,6.25 = 24.19\,\text{min}$$

$$E[S] = E[W] + t_{ce} = 24.19 + 6.25 = 30.44\,\text{min}$$

$$E[X_q] = \lambda_c \cdot E[W] = 8 \cdot \frac{24.19}{60} = 3.225$$

$$E[X] = \lambda_c \cdot E[S] = 8 \cdot \frac{30.44}{60} = 4.059$$

Problem 3:

In this case, the received phone calls cannot be treated as preemptive disruptions to the processing of the various customers through the model applied in the solution of Problem 2, because calls can arrive at any arbitrary timepoint, i.e., not only when some customer is served.

Furthermore, trying to analyze this operation as a classical priority queue with two types of customers, (a) the physically present customers and (b) the incoming phone calls, and further assuming that the phone calls have preemptive priority over the customers, will not work either, because incoming calls do not queue up (i.e. when the clerk answers a phone call cannot receive another one).

A solution to this problem can be synthesized as follows:

(i) To establish the counter stability w.r.t. the processed customers, we need

$$\rho_c = \lambda_c t_c < 1 - u_p \quad (1)$$

where u_p is the clerk utilization for answering phone calls.

Since phone calls have preemptive priority over the customers, and a served phone call blocks any other incoming call, we can get u_p by treating the considered counter as an $M/M/1/1$ queue:

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda_p"] & 1 \arrow[l, bend left, "\mu_p = 1/t_p"] \end{tikzcd}$$

Then $\quad P_0 \lambda_p = P_1 \mu_p \Rightarrow P_1 = \dfrac{\lambda_p}{\mu_p} P_0 = (\lambda_p t_p) P_0$

and $\quad P_0 + \lambda_p t_p P_0 = 1 \Rightarrow P_0 = \dfrac{1}{1+\lambda_p t_p} =$

$$= \frac{1}{1+5\cdot 3/60} = \frac{1}{1.25} = 0.8 \Leftarrow$$

This number is the counter "availability" for processing the incoming customers

Hence, $u_p = P_1 = 1 - P_0 = 0.2$.

Then Eq. (1) becomes:

$$8 \cdot \frac{5}{60} = 0.667 < 1 - 0.2 = 0.8$$

and therefore the considered operation is stable.

(ii) Also, from (i), $TH_c = \lambda_c = 8\ hr^{-1}$.

(iii) To obtain $E[W]$ we work as we did in the computation of this quantity for the basic M/G/1 queue. However, in this case we consider the distribution of the <u>effective service times</u> for the customers, that accounts for the disruption of their service by any incoming calls. Working as in Problem 2, we get the mean and SCV for this distribution as follows:

$$A = \frac{1/\lambda_p}{1/\lambda_p + t_p} = \frac{1/5}{1/5 + 3/60} = 0.8$$

$$t_{c_e} = t_c / A = 5/0.8 = 6.25 \text{ min}$$

$$c_{c_e}^2 = c_c^2 + (1 + c_p^2)\, A(1-A)\, t_p / t_c =$$

$$= \left(\frac{3}{5}\right)^2 + (1+1)\, 0.8(1-0.8)\, 3/5 = 0.552.$$

Next we break down the expected waiting time of an arriving customer as follows:

$$E[W] = E[\text{number of waiting customers upon arrival}] \cdot t_{ce} +$$

$+ \Pr$ [arrival finds a customer in service] $\cdot$

$\cdot E$ [remaining effective service time | customer in service] +

$+ \Pr$ [clerk answers a phone call upon arrival | no customer in service] $\cdot E$ [rem. call time | server answers a call]

(2)

From PASTA,

E [number of waiting customers upon arrival] =

$= E[X_q]$ (3)

and $\Pr$ [arrival finds a customer in service] =

$\lambda_c t_{ce} = 8 \cdot \frac{6.25}{60} = 0.833$ (4)

Also, from Little's law,

$E[X_q] = TH_c \cdot E[W] = \lambda_c E[W]$ (5)

From the memoryless property of the exponential distribution,

E [rem. call time | server answers a call] =

$= t_p$ (6)

while

E [rem. effective service time | customer in service] =

$$= \frac{1 + c_{ce}^2}{2} t_{ce} \qquad (7)$$

On the other hand, the preemptive priority of the phone calls over the customers, together with PASTA, imply that

Pr [clerk answers a call upon arrival | no customer in service] =

= Pr [clerk answers a call upon arrival] =

$= u_p \qquad (8)$

Finally, Equations (2)–(8) imply that

$$E[W] = \lambda_c E[W] t_{ce} + \lambda_c t_{ce} \frac{1 + c_{ce}^2}{2} t_{ce} + u_p t_p \Rightarrow$$

$$\Rightarrow E[W] = \frac{1}{1 - \lambda_c t_{ce}} \left[\lambda_c t_{ce} \frac{1 + c_{ce}^2}{2} t_{ce} + u_p t_p \right]$$

$$= \frac{1}{1 - 0.833} \left[0.833 \, \frac{1 + 0.552}{2} \cdot 6.25 + 0.2 \cdot 3 \right] = 27.78 \text{ min}$$

Also

$$E[S] = E[W] + t_{ce} = 27.78 + 6.25 = 34.03 \text{ min}$$

$$E[X_q] = TH_c \cdot E[W] = 8 \cdot \frac{27.78}{60} = 3.704$$

$$E[X] = TH_c \cdot E[S] = 8 \cdot \frac{34.03}{60} = 4.537.$$

As expected, all these four numbers that characterize the congestion experienced by the incoming customers are higher than the corresponding numbers for Problem 2.

Problem 4:

(i) Every part processed for an order is a Bernoulli trial with success probability $\bar{p}^c = 1-\bar{p}$. Hence, the expected number of trials till success is $1/\bar{p}$. Also, the expected time for all these trials is

$$t_e(\bar{p}) = \left(1/\bar{p}^c - 1\right) \cdot E\,[\text{Proc. time of a failing part}] +$$

$$+ E\,[\text{Proc. time of a good part}] =$$

$$= \left(1/\bar{p}^c - 1\right) \cdot \frac{2}{2} + 2 = \left(\frac{1}{\bar{p}^c} + 1\right) hr \quad (1)$$

We need

$$\lambda t_e(\bar{p}) < 1 \Rightarrow \frac{5}{12} \cdot \left(\frac{1}{\bar{p}^c} + 1\right) < 1 \Rightarrow$$

$$\Rightarrow \frac{1}{\bar{p}^c} < \frac{12}{5} - 1 = 1.4 \Rightarrow \bar{p}^c > 0.714 \Rightarrow$$

$$\Rightarrow p_c \leq 1 - 0.714 = 0.286$$

(ii) Since the shop operation is stable for the considered p value, $TH = \lambda = 5/_{12\,hr} = 0.417\,hr^{-1}$

Also, the utilization is

$$u(p) = \lambda\, t_e(p) = 0.417 \cdot \left(\frac{1}{1-\frac{0.286}{2}} + 1 \right) = 0.9$$

To get the remaining quantities, we also need to compute c_e^2, i.e, the SCV for the effective proc. time per order.

For this computation we define the following r.v.'s:

- T_f = proc. time of a failing part; this follows a uniform distr in $[0,2]$.
- T_p = proc. time of a good part; deterministically equal to 2.
- N = number of proc. parts for an order completion; geometric with success probability $1-p$.
- T_e = effective proc. time for an order

Then,

$$T_e = \sum_{i=1}^{N-1} T_f^{(i)} + T_p \qquad (2)$$

<u>Remark</u>: Eq. (2) was behind the computation of $t_e = E[T_e]$ in part (i).

For $Var[T_e]$ we have:

$$Var[T_e] = Var\left[\sum_{i=1}^{N-1} T_f^{(i)}\right] + Var[T_p] =$$

$$= Var\left[\sum_{i=1}^{N-1} T_f^{(i)}\right] \qquad (3)$$

In (3) we have used the fact that the two terms defining T_e are independent and furthermore T_p is a deterministic quantity.

Also, from the theory of compound r.v.'s (c.f. pages 313-314 in the Primer):

$$Var\left[\sum_{i=1}^{N-1} T_f^{(i)}\right] = Var[T_f] \cdot E[N-1] +$$

$$+ E^2[T_f]\, Var[N-1] =$$

$$Var[T_f]\,(E[N]-1) + E^2[T_f]\,Var[N]$$ (4)

From the uniform distribution of T_f we have:

$$Var[T_f] = \frac{(2-0)^2}{12} = \frac{4}{12} = 0.333$$

Also, from the geometric distribution of N,

$$Var[N] = \frac{1-(1-p)}{(1-p)^2} = \frac{p}{(1-p)^2} = \frac{0.143}{(1-0.143)^2} =$$

$$= 0.195$$

Hence,

$$Var[T_e] = 0.333 \cdot \left(\frac{1}{1-0.143} - 1\right) +$$

$$+ 1 \cdot 0.195 = 0.25$$

Also,

$$c_e^2 = scv[T_e] = \frac{Var[T_e]}{E^2[T_e]} = \frac{0.25}{\left(\frac{1}{1-0.143}+1\right)^2} =$$

$$= 0.053$$

Finally,

$E[S]$ = Expected Order Lead Time =

$$= \frac{1+c_e^2}{2}\frac{u(p)}{1-u(p)}t_e + t_e =$$

$$= \left(\frac{1+0.053}{2}\frac{0.9}{1-0.9}+1\right)\left(\frac{1}{1-0.143}+1\right)$$

$$= 5.7385 \cdot 2.167 = 12.435 \text{ hrs}$$

and

$E[X]$ = Average outstanding orders =

$$= TH \cdot E[S] = \frac{5}{12} \cdot 12.435 = 5.18$$

1	0.833	0.693889	0.57800954	0.48148194	0.40107446	0.33409502	0.27830116	0.23182486	0.19311011	0.16086072
1	1.547	1.798447	1.8621007	1.81102184	1.69414405	1.54371388	1.38051287	1.21751105	1.062413	0.9194236
1	2.38	3.780987	5.01166287	5.98573701	6.68026298	7.10837294	7.30178753	7.29990006	7.14322975	6.86973398