ISYE 7201: Production & Service Systems
Spring 2022
Instructor: Spyros Reveliotis
2nd Midterm Exam (Take Home)
Release Date: March , 2022
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While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

Please, send me your responses as a pdf file attached to an email. Name the pdf file by your last name (only). The pdf file can be a scan or photos of a hand-written document, but, please, write your answers clearly and thoroughly. Also, make sure that the pdf file is not too big; you can reduce the size of your file by loading it into Adobe Acrobat and saving it with the "reduced" size option before emailing it to me.

Finally, report any external sources (other than your textbook) that you referred to while preparing the solutions.

Problem 1 (20 points): Consider a Continuous-Time Markov Chain (CT-MC) that cycles among 100 states, numbered 1, 2, 3, ..., 100; i.e., the chain goes from state 1 to state 2 to state 3, etc., all the way up to state 100, and from there back to state 1, and repeats this cycle. The sojourn time at each state is exponentially distributed with rate μ. Argue that this CT-MC is ergodic, and compute the limiting distribution.

Problem 2 (20 points): A single repairperson looks after two machines $M1$ and $M2$. Each time it is repaired, machine Mi stays up for an exponential time with rate $\lambda_i,\ i = 1, 2$. When machine Mi fails, it requires an exponentially distributed amount of work with rate μ_i to complete its repair. In the case that both machines are down, the repairperson can determine how to split his/her time between the repair of the two machines. Also, when machine Mi is up, it generates revenue with rate r_i. Determine how the repairperson must split his/her time between the two machines, when both of them are down, in order to maximize the expected total revenue rate generated by these two machines.

Problem 3 (20 pts) Consider a CT-MC $\{X(t),\ t \geq 0\}$ with infinitesimal generator R, and further assume that the embedded DT-MC $\{\hat{X}_k,\ k \geq 0\}$ is irreducible and positive recurrent. Show that if there exists a row vector $\mathbf{p} > \mathbf{0}$ (component-wise) that is a solution to the system of equations

$$\mathbf{p}R = \mathbf{0} \;\; ; \;\; \sum_i p_i = 1.0$$

then, the CT-MC $X(t)$ is ergodic.

Problem 4 (20 pts): Consider an inventory system where customers arrive according to a Poisson process with rate λ, and each customer poses a random demand of D units, with $D \in \{1, 2, \ldots, r+1\}$; hence, the corresponding probabilities satisfy $p_i > 0, \forall i = 1, \ldots, r+1$; $\sum_{i=1}^{r+1} p_i = 1.0$.

The inventory is managed according to the following (Q, r) policy: It is continuously monitored, and whenever it drops below $r + 1$, there is an immediate replenishment of Q units; in the corresponding terminology, r is the reorder point (ROP). Also, assume that $Q \geq r+1$. Finally, also suppose that the system starts empty at time $t = 0$.

i. (5 pts) Show that the operation of this inventory system for $t > 0$ can be modeled by a CT-MC with state space $S = \{r + 1, \ldots, r + Q\}$.

ii. (5 pts) Argue that the CT-MC defined in item (i) above is ergodic.

iii. (10 pts) Show that the limiting distribution for the considered CT-MC is *uniform*.

Remark: The uniformity of the limiting distribution is a remarkable result. Also, notice that even though the assumption of instantaneous replenishment might seem unrealistic, the considered model obtains practical relevance if we assume that r is defined w.r.t. the *inventory position*, which also accounts for the outstanding replenishment orders, and not only the *on-hand-inventory*.

Problem 5 (20 points) People access a slot machine according to a Poisson process with a rate $\lambda = 10$ persons per hour. Each person drops 25 cents in the machine, and the machine returns one dollar with probability $p = 0.1$ or nothing with the remaining probability. At the beginning of the day, we place 20 dollars in the machine.

i. **(10 pts)** What is the expected amount of money in the machine after 6 hours?

ii. **(10 pts)** What is the probability that the amount of money in the machine will have been doubled in 6 hours?

Remark: Here is also a result that can be useful in the solution of the last problem (S. M. Ross, "Stochastic Processes", 2nd ed., pgs 342-343):

Consider a random walk $S_n = \sum_{i=1}^{n} X_i,\ n \geq 1$, with $E[X] \neq 0$, and further assume that there exists $\theta \neq 0$ s.t. $E[e^{\theta X}] = 1.0$. Then, for any given $A, B > 0$, the probability that the random walk reaches a value greater than or equal to A before it reaches a value less than or equal to $-B$, is approximately equal to

$$\frac{1 - e^{-\theta B}}{e^{\theta A} - e^{-\theta B}}$$

Finally, please, explain clearly all your answers.

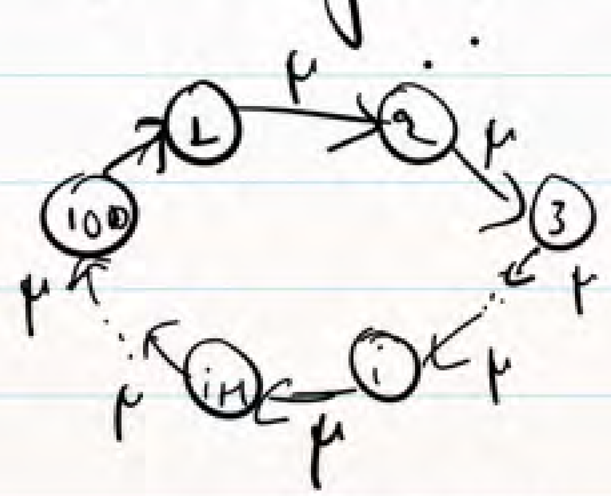
ISYE 7201 - SPRING 2022

MIDTERM II SOLUTIONS

Problem 1

The embedded DTMC is finite-state and irreducible. Hence, it is also positive recurrent. Furthermore, the finiteness of the state space implies that the mean recurrence time of every state in the dynamics of the CTMC is finite, as well (Why?). Hence, according to the corresponding theorem that was presented in the lectures, the considered CTMC is ergodic.

The state transition diagram (STD) of this CTMC is as follows:

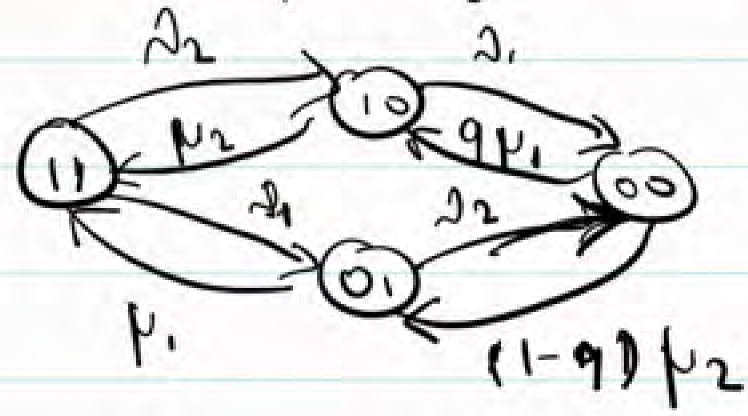


This STD is completely symmetric w.r.t. the role of every state in it. The limiting distribution p must reflect this symmetry, and therefore, p is a <u>uniform</u> distribution; i.e.,

$$p_i = 1/100, \quad \forall i \in \{1, \ldots, 100\}$$

Problem 2

The operation of the considered facility can be modeled as a CTMC with state
$$(x_1, x_2), \quad x_i \in \{0,1\}, \ \forall i \in \{1,2\},$$
where x_i denotes the operational status of machine M_i ($x_i = 1 \Rightarrow$ up ; $x_i = 0 \Rightarrow$ down).
The corresponding STD is as follows:



The parameter q in the above model is the percentage of his/her time that the repair person works on machine M_1 when both machines are down (i.e., in this case, the repair person works in a time-sharing mode on both machines).

Since this CTMC is finite-state and irreducible, it is ergodic. Let p_{ij} denote the limiting probability of state (i,j). These probabilities must satisfy the following system of equations:

$$\begin{cases} P_{11}(\lambda_1+\lambda_2) = \mu_2 P_{10} + \mu_1 P_{01} \\ P_{10}(\lambda_1+\mu_2) = \lambda_2 P_{11} + q\mu_1 P_{00} \\ P_{01}(\lambda_2+\mu_1) = \lambda_1 P_{11} + (1-q)\mu_2 P_{00} \\ P_{11}+P_{10}+P_{01}+P_{00} = 1 \end{cases} \iff .$$

$$\underbrace{\begin{bmatrix} \lambda_1+\lambda_2 & -\mu_1 & -\mu_2 & 0 \\ -\lambda_2 & \lambda_1+\mu_2 & 0 & q\mu_1 \\ -\lambda_1 & 0 & \lambda_2+\mu_1 & (1-q)\mu_2 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} P_{11} \\ P_{10} \\ P_{01} \\ P_{00} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{b}$$

From Crammer's rule,

$$\forall (i,j), \quad P_{ij} = \frac{|A(i,j)|}{|A|}$$

where $A(i,j)$ is the matrix obtained from A by replacing the column of A corresponding to P_{ij} with vector b.

It is easy to see that $|A|$ and each $|A(i;j)|$ are linear functions of q, with the exception of $A(0,0)$ which is constant.

Furthermore, for any $q \in [0,1]$, the average reward rate is

$$R(q) = (r_1 + r_2) P_{11}(q) + r_1 P_{10}(q) + r_2 P_{01}(q)$$

Hence,

$$R(q) = \frac{aq + b}{cq + d}$$

where a, b, c and d are constants determined by the above equations providing $P_{ij}(q)$ and $R(q)$.

But $$\frac{dR(q)}{dq} = \frac{a(cq+d) - c(aq+b)}{(cq+d)^2} =$$

$$= \frac{ad - cb}{(cq+d)^2}$$

This result further implies that

(i) if $ad - cb > 0$ then $q = 1$

(ii) if $ad - cb < 0$ then $q = \emptyset$

(iii) if $ad - cb = 0$ then q can be any value in $[0, 1]$.

Problem 3

Since the embedded DT-MC $\hat{X}_k$ is irreducible and positive recurrent, according to Theorem 1 that characterizes the limiting behavior of CT-MCs in the Primer, to establish the ergodicity of the CT-MC $X(t)$, it suffices to show that

$$\forall j, \quad E[T_{jj}] < \infty \qquad (1)$$

where T_{jj} is the recurrence time for state j.
Next, we establish this result.

First, starting from the equation

$$\forall i, \quad p \cdot R[:, i] = 0 \qquad (2)$$

and reversing the steps that led to this equation in the derivation of the second approach for the computation of the limiting distribution p that was presented in the Primer (c.f. pgs 106-107), we get

$$\forall i, \quad \frac{p_i}{\tau_i} = \sum_{j \neq i} \frac{p_j}{\tau_j} \hat{P}_{ji} \qquad (3)$$

When combined with the irreducibility and the positive recurrence of $\hat{X}_k$, Eq. (3) implies that

$$\forall i, j \quad \pi_j / \pi_i = (p_j/\tau_j) / (p_i/\tau_i) = \tau_i p_j / \tau_j p_i \qquad (4)$$

where $(\pi_1, \pi_2, \ldots, \pi_i, \ldots)$ is the stationary distribution of $\hat{X}_k$.

Eq. (4) can be rewritten as

$$\forall i, j, \quad \frac{p_i}{p_j} = \frac{\pi_i \tau_i}{\pi_j \tau_j}$$

which further implies that

$$\forall i, \quad \frac{p_i}{\sum_j p_j} = \frac{\pi_i \tau_i}{\sum_j \pi_j \tau_j} \qquad (5)$$

Eq. (5), combined with the fact that $\sum_j p_j = 1$, imply that

$$\forall i, \quad \sum_j \pi_j \tau_j = \frac{\pi_i \tau_i}{p_i}$$

and therefore $\quad \sum_j \pi_j \tau_j < \infty \qquad (6)$

Then, Eq. (1) above results from Eq. (6) and the identity of Eq. (2) in pg. 104 in the Primer.

Problem 4

(i) Let $\{X(t), t \geq 0\}$ denote the inventory position. We have $X(0) = 0$, and therefore at $t = 0^+$, there is a replenishment order for Q units which brings $X(t)$ in S, since $Q \geq r+1$.

It is also clear that $X(t)$ is constant between two customer arrivals, and this time interval is exponentially distributed with rate λ.

Next, let $\hat{X}_k$ denote the state of $X(t)$ right after the service of the k-th customer, for $k \geq 1$. Then, it is easy to see that

$$\forall k \geq 1, \quad \hat{X}_k = \hat{X}_{k-1} - D_k + Q \cdot I_{\{\hat{X}_{k-1} - D_k \leq r\}} \quad (1)$$

In the above recursion it is assumed that $\hat{X}_0 = Q$, for the reasons explained in the opening paragraph, and this assumption, together with the provided distribution for D and the fact that $Q \geq r+1$, further imply that $\hat{X}_{k-1} - D_k \geq 0$.

Hence, $\hat{X}_k \in S$, $\forall k \geq 1$.

It is clear from (1) that $\{\hat{X}_n, n \geq 1\}$ is a DT-MC evolving in S, and the one-step transition probabilities for this MC are specified from the distribution of D and the applied ordering policy.

Furthermore, from the opening remarks for the stochastic process $\hat{X}(t)$, it is clear that $\hat{X}_n$ constitutes an embedded DT-MC for $X(t)$. And since the state sojourn times of $X(t)$ are exponentially distributed, $X(t)$ is a CT-MC.

(ii) From the corresponding theorem in the Primer (c.f. pg. 92) and the fact that the underlying state space S is finite, to establish the ergodicity of $X(t)$, it suffices to establish the irreducibility of $\hat{X}_n$.

The positivity of $p_1 = \text{Prob}(D=1)$ and the adopted ordering policy imply that every state $r+i$, $i=1,\ldots,Q-1$, is accessible from state $r+Q$. In addition, $p_1 > 0$ and the ordering policy imply that state $r+Q$ is accessible from state $r+1$.

Hence, $\hat{X}_k$ is irreducible.

(iii) In order to establish the sought result, first notice that the considered CT-MC $X(t)$ is <u>naturally uniformized</u> by λ. Hence, its limiting distribution is equal to the limiting distribution of its embedded DT-MC $\hat{X}_k$. Let $\pi = (\pi_{r+1}, \pi_{r+2}, \ldots, \pi_{r+Q})$ denote the last distribution. Next we shall show that π is uniform, i.e.,

$$\pi_i = \frac{1}{Q}, \quad \forall\, i \in \{r+1, \ldots, r+Q\}$$

Since $\hat{X}_k$ is irreducible and finite-state, π exists and it is the unique solution to

$$\pi = \pi \cdot \hat{P} \quad (2) \quad ; \quad \sum_i \pi_i = 1 \quad (3)$$

where $\hat{P}$ is the corresponding one-step transition prob. matrix. In more conceptual terms, each scalar equation generated by (2) implies that the corresponding π_i is equal to the sum $\sum_{j \neq i} \pi_j \hat{p}_{ji}$, where $\hat{p}_{ji}$ is the probability of transitioning from state j to state i in one step.

Then, to establish the claimed uniformity of Π, it suffices to show that the total probability of transitioning into state i in one step, $\sum_{j \neq i} \hat{p}_{ji}$, is equal to 1.0 for any state $i \in \{r+1, \ldots, r+Q\}$.

We proceed to establish this result by distinguishing two distinct groups of states.

I) <u>State $s \in \{Q, Q+1, \ldots, Q+r\}$</u>

Each state $Q+i$, $i=0, \ldots, r$, in the above set, is accessible from the higher states $Q+i+j$, $j=1, \ldots, r-i$, with corresponding probabilities p_j (i.e, upon the occurrence of a demand $D=j$).

Also, state $Q+i$ is accessible from states $r+1, r+2, \ldots, r+1+i$ with corresponding probabilities $p_{r+1-i}, p_{r+2-i}, \ldots, p_{r+1}$.

And all the aforelisted states are legitimate (i.e., they belong in S) since $i \in \{0, 1, \ldots, r\}$ and $Q \geq r+1$.

Then, summing up the total transition probability in any state $Q+i$, $i=0,1,\dots,r$, we get

$$\sum_{j=1}^{r-i} p_j + \sum_{j=1}^{1+i} p_{r-i+j} =$$

$$= (p_1 + p_2 + \dots + p_{r-i}) + (p_{r-i+1} + p_{r-i+2} + \dots + p_{r+1}) =$$
$$= 1.$$

II) If $Q > r+1$, we also need to consider the states $s \in \{r+1, r+2, \dots, Q-1\}$.

For any such state, the only possible transitions into the state are from the $r+1$ states that immediately succeed this state in S. Also, the corresponding probabilities are $p_1, p_2, \dots, p_{r+1}$, and therefore the total

transition probability into the state is

$$\sum_{i=1}^{r+1} p_i = 1$$

Hence, the claim has been established.

Problem 5

Let $\{W(t),\ t \geq 0\}$ be the stochastic process representing the amount of money in the slot machine at time t. Then, $W(0) = 20$, and from the problem description we can see that the state space of this process is $S = \{0, 0.25, 0.5, 0.75, 1.0, 1.25, \ldots\}$.

Furthermore, state 0 is absorbing, and from every other state $s \in S$, the process transitions to a new s' upon the arrival of a new customer as follows:

$$(1) \qquad \hat{P}_{ss'} = \begin{cases} 0.9 & \text{if } s' = s + 0.25 \\ 0.1 & \text{if } s' = \max\{0, s - 0.75\} \\ 0 & \text{o.w.} \end{cases}$$

Also, the sojourn time for a visit to any state $i \neq 0$ is exponentially distributed with rate $\lambda = 10\ \text{hr}^{-1}$.

From the above discussion, it is clear that $W(t)$ is a CT-MC with its embedded DT-MC $\{\hat{W}_k,\ k \geq 0\}$ defined by (1), and a common sojourn time distribution for all states $i \neq 0$.

Both of the problem questions concern the transient behavior of $W(t)$. To answer them efficiently, it is convenient to establish first that the probability of the process absorption in state 0 over the considered time interval of 6 hours is negligible. For this, we shall use the result in the Remark that accompanies this problem in the exam.

So, first notice that for the embedded DT–MC $\hat{W}_k$, we have

$$\forall k \geq 0, \ \forall s_k \notin \{0, 0.25, 0.5\}, \qquad \hat{W}_{k+1} = \hat{W}_k + X_k$$

with

$$X_k = \begin{cases} 0.25 & \text{w.p. } 0.9 \\ -0.75 & \text{w.p. } 0.1 \end{cases} \qquad (2)$$

Hence, until a potential absorption to state $\emptyset$, $\hat{W}_k$ constitutes a random walk with its increments X_k distributed according to Eq. (2) above.

We also have

$$E[X_k] = 0.9 \cdot 0.25 + 0.1(-0.75) = 0.15, \quad (3)$$

and therefore, the considered random walk is transient, diverging to $+\infty$.

In order to apply the provided result in the Remark, we also need $\theta \neq 0$, s.t.

$$0.9e^{0.25\theta} + 0.1e^{-0.75\theta} = 1 \qquad (4)$$

Setting $e^{\theta} = y$ and $y^{1/4} = z$, (4) becomes

$$0.9z + 0.1z^{-3} = 1 \Leftrightarrow 0.9z^4 - z^3 + 0.1 = 0 \qquad (5)$$

Since we need $\theta \neq 0$, we also need $z \neq 1$.
It can be verified that $z \simeq 0.6$ solves (5) pretty accurately. Hence,

$$e^{\theta} = 0.6^4 \Rightarrow \theta = 4\ln 0.6 \simeq -2.0433 \qquad (6)$$

It is also more convenient to consider the random walk $\tilde{W}_k$ with $\tilde{W}_0 = 0$ and increments $\tilde{X}_k = -X_k$.
Then $E[\tilde{X}_k] = -E[X_k] < 0$
and

$$E[e^{\tilde{\theta}\tilde{X}_k}] = 1 \Leftrightarrow E[e^{-\tilde{\theta}X_k}] = 1$$

which is satisfied by setting $\tilde{\theta} = -\theta = 2.0433$ (7)

In the dynamics of $\tilde{W}_k$, the bankruptcy of the considered operation is modelled by the event $\tilde{W}_k \geq 20$. Then, setting $A = 20$, $B = -\infty$, and using the obtained $\tilde{\theta}$, the result in the provided Remark gives

$$P(\text{Bankruptcy}) \simeq e^{-\tilde{\theta} A} =$$

$$= e^{-2.0433 \cdot 20} \simeq 1.79 \times 10^{-18} \qquad (8)$$

Since the above probability is higher that the probability of going bankrupt within 6 hours, we have established our aforestated objective, and we are ready to answer the problem questions.

(i) To answer this part, let $\{N(t), t \geq 0\}$ denote the Poisson process tracing the customer arrivals in the interval $(0, t]$, and also consider the counting processes $\{N_W(t), t \geq 0\}$ and $\{N_\ell(t), t \geq 0\}$ that count, respectively, the winners and the losers in the same interval. Clearly, the last two processes are obtained from $N(t)$ by Bernoulli splitting with corresponding probabilities p and $1-p$, and therefore, they are Poisson processes with respective rates

λp and $\lambda(1-p)$. They are also independent from each other.

From the above remarks and the fact that bankruptcy is negligible, we also have:

$$(9)\quad \forall t,\ W(t) = W(0) + 0.25\, N(t) - 1.0\, N_w(t)$$
$$\Rightarrow E[W(t)] = W(0) + 0.25\, E[N(t)] - 1.0\, E[N_w(t)]$$
$$\Rightarrow E[W(t)] = W(0) + 0.25\,(\lambda t) - 1.0\,(\lambda p t)$$
$$\Rightarrow E[W(6)] = 20 + 0.25\,(10\cdot 6) - 1.0\,(10\cdot 0.1\cdot 6)$$
$$= 29\ (\$)$$

(ii) We want to compute

$$P[W(6) \geq 40] =$$
$$= \sum_{n=0}^{\infty} P[W(6) \geq 40 \mid N(6) = n]\, P[N(6) = n]$$
$$= \sum_{n=0}^{\infty} P[W(6) \geq 40 \mid N(6) = n]\, e^{-60}\frac{60^n}{n!} \quad (10)$$

In Eq. (9), we have used the fact that $N(6) \sim \text{Poisson}(10\cdot 6)$.

We also have from (9) that

$$P[W(6) \geq 40 \mid N(6) = n] \neq 0 \iff$$

$$\iff \exists n_W \in \mathbb{Z}_0^+ \text{ s.t. } 20 + 0.25n - n_W \geq 40$$

$$\iff \exists n_W \in \mathbb{Z}_0^+ \text{ s.t. } n_W \leq 0.25n - 20 \qquad (11)$$

Hence,

$$P[W(6) \geq 40 \mid N(6) = n] \neq 0 \implies$$

$$\implies 0.25n - 20 \geq 0 \implies n \geq 80. \qquad (12)$$

For any such n,

$$P[W(6) \geq 40 \mid N(6) = n] =$$

$$= \sum_{i=0}^{\lfloor 0.25n-20 \rfloor} \binom{n}{i} 0.1^i \cdot 0.9^{n-i} \qquad (13)$$

Finally, from (10), (12) and (13) we have:

$$P[W(6) \geq 40] = e^{-60} \sum_{n=80}^{\infty} \frac{60^n}{n!} \sum_{i=0}^{\lfloor 0.25n-20 \rfloor} \binom{n}{i} 0.1^i \cdot 0.9^{n-i} \qquad (14)$$

Organizing the above computation in Excel and truncating the outer summation at $n = 110$, we got $\underline{P[W(6) \geq 40] \simeq 6.19 \times 10^{-6}}$.