ISYE 7201: Production & Service Systems
Spring 2022
Instructor: Spyros Reveliotis
Final Exam (Take Home)
Release Date: April 27, 2022
Due Date: May 3, 2022

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

Please, send me your responses as a pdf file attached to an email. Name the pdf file by your last name (only). The pdf file can be a scan or photos of a hand-written document, but, please, write your answers clearly and thoroughly. Also, make sure that the pdf file is not too big; you can reduce the size of your file by loading it into Adobe Acrobat and saving it with the "reduced" size option before emailing it to me.

Finally, report any external sources (other than your textbook) that you referred to while preparing the solutions.

Problem 1 (20 points) Please, answer the following questions:

i. (10 pts) Provide an example showing that the throughput function $TH(N)$ defined in the theory of closed queueing networks presented in class (c.f. page 215 of the Primer) may be decreasing as the number of circulating customers increases from some value N to $N+1$.

ii. (10 pts) Reconsider the question in part (i) for the special case where the processing rates at the network workstations are constant (i.e., $\forall i \in \{1,\ldots,J\} : \ \mu_i(n) = \mu_i, \ \forall n > 0$). Is $TH(N)$ a nondecreasing function in this case? You need to either prove this property under the considered assumption, or provide a counter-example.

Problem 2 (20 points) Consider a Gordon-Newell network with J single-server workstations, $WS_j, \ j = 1,\ldots,J$, where a customer leaving workstation WS_j can move to any workstation of the network, including station j itself, with equal probability. Service times at workstation WS_j are exponentially distributed with rate $\mu_j = \mu$, for all $j = 1,\ldots,J$.

i. (10 pts) Perform a mean value analysis (MVA) for this network when there are N customers circulating in it; in particular, compute (i) the throughput $TH_j(N)$ of each workstation WS_j, (ii) the expected sojourn time $S_j(N)$ of a customer during a single visit at this station, (iii) the expected waiting time in the station queue during a single visit, $W_j(N)$, and (iv) the average number of customers at this station, $L_j(N)$, at equilibrium.

ii. (5 pts) Compute $\lim_{N\to\infty} TH_j(N), \ j = 1,...,J$, from your results in part (i). Also, provide an intuitive explanation of these limits.

iii. (5 pts) Reconsider the previous two parts for the case where a customer moving out from some workstation WS_j cannot re-enter station j upon this transition. Are there any interesting conclusions to be drawn from the comparison of the results for the two considered cases?

Hint: Exploit the symmetries implied by the problem definition.

Problem 3 (20 pts – a "malevolent" queue :) Consider a facility that serves two types of customers, 1 and 2, according to the non-preemptive priority queueing model discussed in the lectures. Furthermore, waiting customers are charged at a rate of C_i dollars per time unit for type i customer,

$i = 1, 2$ (this could be, for instance, the corresponding rates that each customer type is charged for using the parking lot of the facility). Show that in order to maximize the expected proceeds from these charges, the company should give priority to the customer type that maximizes the ratio τ_{p_i}/C_i, where τ_{p_i} denotes the mean service time for type i customers.

Problem 4 (20 pts) A single-server workstation processes lots of parts that arrive according to a Poisson process with rate 5 lots per hour. Each lot can consist of 2 to 5 parts and the lot size is uniformly distributed. Lots are processed on a First-Come-First-Serve basis, and a lot that is brought to the server for processing, has its parts processed one by one, with the part processing times following a normal distribution with mean 3 min and st. deviation 1 min. But all parts belonging in a lot eventually leave the station together, as a single lot.

Please, answer the following questions:

i. (10 pts) Show that this workstation is stable, and perform a Mean Value Analysis (MVA) for it.

ii. (10 pts) Assume that the part processing rate r_p can vary in the range of [15, 25] parts per hour, while the corresponding-part-processing time distributions remain normal with the same st. deviation of 1 min. On the other hand, the processing cost per part is an increasing linear function of r_p, $f_p(r_p)$, with $f_p(15) = 2$ and $f_p(25) = 5$. Also, it is estimated that a minute spent by a part at this workstation translates into a cost of 5 cents. Based on this information, find an optimal part processing rate for this workstation, assuming that this rate can be fixed up to its first decimal point.

Problem 5 (20 points) Consider a manufacturing workstation that processes two types of parts and the station server is an industrial furnace. Each part type is processed separately by the furnace in batches of 5 units per batch. Formed batches from both part types are waiting for processing in a common queue, and they are processed in a FCFS mode. The part arrival rates, the SCV's for the part inter-arrival times, and the batch processing times for each part type are as follows:

Part type	$\lambda(i)$	$c_a^2(i)$	$t_p(i)$	Batch Proc. Time Distribution
1	5	1.5	0.5	Deterministic
2	4	2.2	0.35	Deterministic

Please, answer the following questions.

i. (10 pts) Show that the considered workstation is stable and perform a mean value analysis (MVA) for each part type.

ii. (10 pts) Compute a batch size k^*, common for both part types, that minimizes the expected number of batches waiting for processing at this workstation. Also, provide a natural explanation of your solution, and discuss its implications for the pursued objective.

Hint: For the computation of the required statistics of the batch arrival process to the common common queue refer to pages 315-316 in the Course Notes.

Finally, please, explain clearly all your answers.

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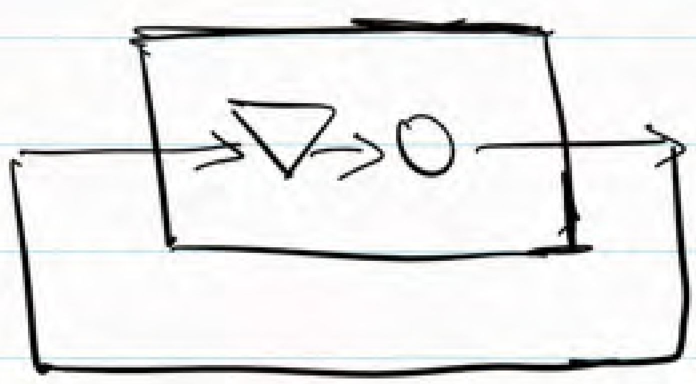
FINAL EXAM SOLUTIONS

Problem 1

(i) Consider the following single-station CQN:



Furthermore, assume that $\mu_1(1) = 1$ and $\mu_1(2) = 1/2$.
Then, it is clear that for this station
$TH_1(1) = 1$ and $TH_1(2) = 1/2$.
Also, setting $v_1 = 1$, we get that
$TH(1) = 1$ and $TH(2) = 1/2$

(ii) We prove that, in this case, $TH(N)$ is non-decreasing in N using a sample-path-based argument.
For this, first notice that when the proc. rates μ_i are constant for each station i, a sample

path for the network is specified completely by providing: (i) an initial distribution of the N jobs at the system workstations; (ii) a sequence of proc. times for each server, drawn from the corresponding exp. distr.; and (iii) a sequence of routings for the jobs completed at each server, specified according to the corresponding routing distr.

Pick a set of sequences of proc. times for the servers of the network and a set of routing sequences for the completed jobs at each server. Also consider an initializing distribution of the network when it is run with $N-1$ jobs, an another initialization where there is an extra job at one of the workstations. Mark this job, and run the network with the predetermined data but avoiding to use the marked job until the station where it resides becomes empty of any job.

Notice that up to this time point, the entire network runs exactly in the same manner with the network with only $N-1$ (unmarked) jobs.

On the other hand, the availability of the marked job when its hosting workstation gets empty of other jobs, expedites the execution of the job proc. sequence at the server of this station without impeding the processing of the other servers.

Furthermore, when the marked job reaches its next hosting station, it behaves as it did in the previous station; i.e., it waits until this station gets empty and only then it occupies the server.

Hence it is easy to see that such an operation is compatible with the presumed dynamics of the considered network, and it can only expedite the processing to take place at each server, compared to the timing of this processing in the execution of the considered sample path with N-1 jobs.

Since the sample path data (i.e., the service times, job routings, and the initial distribution of jobs at the system stations) are quite arbitrary, the above argument implies that the throughput resulting from running the network with a given set of sample-path data and N-1 jobs can only improve by adding an extra job. —

Remark: But it is possible that the throughput will not increase with an increase of N.
For a concrete example, consider again the single-station network of the first part with a constant proc. rate μ. It is clear that the marked job in the previous proof will never be used by the server of its hosting station in this case.

Remark: The previous intends to give the basic structure of the overall argument. It can be further formalized by introducing all the necessary notation for the involved quantities and their relationship, as defined by the network structure and dynamics.

Remark: Clearly, the above proof also holds for the case where proc. rates $\mu_i(x_i)$ are monotonically increasing with x_i, at each station.

Remark: Interestingly, while working on this problem, Moyi was able to trace the above proof in the following paper:

* Ivo Adan, Jan Van der Wal, "Monotonicity of the Throughput of a Closed Queueing Network in the Number of Jobs", Operations Research 37 (6): 953-957, 1989.

The authors of this paper indicate that this proof was given to them as an alternative proof for their main result by a referee of that paper; c.f. Section 5 in it.

Remark: Sample-path based proofs can be quite insightful (and, as in this case, pretty simple, as well) because they look directly at the dynamics of the underlying system, almost as if you were watching (tracing) a simulation of it.

Next, I also provide a more algebraic proof for the result of part (ii) that has a more algebraic flavor, and it is based on the definition of TH(N) that we discussed in class. This proof was presented by Daan.

The proof is based on an induction on the number of workstations J.

Base Case: for $J=1$, we have

$$P(|Y|=N) = P(Y_1=N) = P(Y_1=0)\left(\frac{v_1}{\mu_1}\right)^N.$$

Therefore,

$$TH(N) = \frac{P(|Y|=N-1)}{P(|Y|=N)} = \frac{\mu_1}{v_1}$$

Hence, $TH(N)$ is constant for any $N \geq 1$ in this case, for any selection of v_1.

Inductive Step: We assume that $TH(N)$ is non-decreasing for any CQN satisfying the problem assumptions with up to $J-1$ workstations, and we shall show that $TH(N)$ must be non-decreasing for such CQNs with J workstations.

In the following, we index the various quantities with the number of workstations in the corresponding CQNs. This index appears as a superscript to these quantities. Then we have:

$$P^J(|Y|=N) = \sum_{|x|=N} \prod_{j=1}^{J} P(Y_j=0)\left(\frac{v_j}{\mu_j}\right)^{x_j} =$$

$$= \sum_{x_J=0}^{N} P(Y_J=0)\left(\frac{v_J}{\mu_J}\right)^{x_J} \sum_{\substack{|\mathbf{x}_{-J}|=\\=N-x_J}} \prod_{j=1}^{J-1} P(Y_j=0)\left(\frac{v_j}{\mu_j}\right)^{x_j} =$$

$$= \sum_{x_J=0}^{N} P(Y_J=0)\left(\frac{v_J}{\mu_J}\right)^{x_J} P^{J-1}\left(|Y_{-J}| = N-x_J\right) \quad (1)$$

W.l.o.g, we can assume that $\frac{v_J}{\mu_J}=1$ (just pick $v_J=\mu_J$, and this selection also settles the value of every other v_j, $j=1,\dots,J-1$).

Then (1) implies that:

$$TH^J(N) = \frac{P^J(|Y|=N-1)}{P^J(|Y|=N)} = \frac{\sum_{x=0}^{N-1} P^{J-1}(|Y_{-J}|=x)}{\sum_{x=0}^{N} P^{J-1}(|Y_{-J}|=x)}$$

and

$$TH^J(N+1) - TH^J(N) =$$

$$= \frac{\sum_{x=0}^{N} P^{J-1}(|Y_{-J}|=x)}{\sum_{x=0}^{N+1} P^{J-1}(|Y_{-J}|=x)} - \frac{\sum_{x=0}^{N-1} P^{J-1}(|Y_{-J}|=x)}{\sum_{x=0}^{N} P^{J-1}(|Y_{-J}|=x)} \quad (2)$$

From (2) it follows that in order to show that

$$TH^J(N+1) \geq TH^J(N) \quad (3)$$

it suffices to show that:

$$\left[\sum_{x=0}^{N} P^{J-1}(|Y-J|=x)\right]^2 \geq$$

$$\geq \left[\sum_{x=0}^{N-1} P^{J-1}(|Y-J|=x)\right]\left[\sum_{x=0}^{N+1} P^{J-1}(|Y-J|=x)\right] \Leftrightarrow$$

$$\Leftrightarrow \left[\cancel{\sum_{x=0}^{N-1} P^{J-1}(|Y-J|=x)} + P^{J-1}(|Y-J|=N)\right] \cdot$$

$$\cdot \left[\sum_{x=0}^{N} P^{J-1}(|Y-J|=x)\right] \geq$$

$$\geq \left[\sum_{x=0}^{N-1} P^{J-1}(|Y-J|=x)\right]\left[\cancel{\sum_{x=0}^{N} P^{J-1}(|Y-J|=x)} +\right.$$

$$\left. + P^{J-1}(|Y-J|=N+1)\right] \Leftrightarrow$$

$$\Leftrightarrow P^{J-1}(|Y-J|=N)\left[\sum_{x=0}^{N} P^{J-1}(|Y-J|=x)\right] \geq$$

$$\geq P^{J-1}(|Y-J|=N+1)\left[\sum_{x=0}^{N-1} P^{J-1}(|Y-J|=x)\right] \Leftrightarrow$$

$$\Leftrightarrow P^{J-1}(|Y-J|=N) \cdot P^{J-1}(|Y-J|=0) +$$

$$+ P^{J-1}(|Y-J|=N)\left[\sum_{x=1}^{N} P^{J-1}(|Y-J|=x)\right] \geq$$

$$\geq P^{J-1}(|Y-J|=N+1)\left[\sum_{x=0}^{N-1} P^{J-1}(|Y-J|=x)\right] \Leftrightarrow$$

$$\Leftrightarrow \; P^{J-1}(|Y_{-J}|=N)\, P^{J+1}(|Y_{-J}|=0) \; +$$

$$+ \; P^{J-1}(|Y_{-J}|=N)\left[\sum_{x=0}^{N-1} P^{J-1}(|Y_{-J}|=x+1)\right] \geq$$

$$\geq P^{J-1}(|Y_{-J}|=N+1)\left[\sum_{x=0}^{N-1} P^{J-1}(|Y_{-J}|=x)\right] \qquad (4)$$

Since the first term in the left-hand-side of (4) is nonnegative, to establish the validity of this inequality, it suffices to show that:

$$\forall x = 0, 1, \dots, N-1,$$

$$P^{J-1}(|Y_{-J}|=N)\, P^{J-1}(|Y_{-J}|=x+1) \geq$$

$$\geq P^{J-1}(|Y_{-J}|=N+1)\, P^{J-1}(|Y_{-J}|=x) \; \Leftrightarrow$$

$$\Leftrightarrow \; \frac{P^{J-1}(|Y_{-J}|=N)}{P^{J-1}(|Y_{-J}|=N+1)} \geq \frac{P^{J-1}(|Y_{-J}|=x)}{P^{J-1}(|Y_{-J}|=x+1)} \; \Leftrightarrow$$

$$\Leftrightarrow \; TH^{J-1}(N+1) \geq TH^{J-1}(x+1) \qquad (5)$$

Eq. (5) is true because of the induction hypothesis, and therefore, (3) is true. —

Problem 2

(i) The symmetry exhibited by this network w.r.t. the role of the various workstations in it, implies that $v_1 = v_2 = \dots = v_J$. So, set $v_j = 1$, $\forall j$. Next consider an arrival at some station j of this network when the network is at equilibrium. The aforementioned symmetry together with the arrival theorem and the memoryless property of the exponential distribution imply that

$$\forall j, \quad S_j(N) = \frac{N-1}{J}\left(\frac{1}{\mu}\right) + \frac{1}{\mu} \qquad (1)$$

Also, from Little's law:

$$\forall j, \quad L_j(N) = v_j TH(N) S_j(N) = TH(N) S_j(N)$$

and since $\sum_{j=1}^{J} L_j(N) = N$, we have

$$TH(N) \sum_{j=1}^{J} S_j(N) = N \implies$$

$$\implies TH(N) = N \Big/ \sum_{j=1}^{J} S_j(N) \qquad (2)$$

Also,
$$\sum_{j=1}^{J} S_j(N) = J\left(\frac{N-1}{J}\frac{1}{\mu}+\frac{1}{\mu}\right) =$$
$$= (N-1)\,{}^1\!/\!_\mu + {}^J\!/\!_\mu .$$

Furthermore,
$$\forall j, \quad TH_j(N) = v_j\, TH(N) =$$
$$= \frac{N}{(N-1)\frac{1}{\mu}+\frac{J}{\mu}} = \frac{N}{N+J-1}\mu \qquad (3)$$

Finally,
$$\forall j, \quad L_j(N) = \left(\frac{N}{N+J-1}\mu\right)\left(\frac{N-1}{J}\frac{1}{\mu}+\frac{1}{\mu}\right) =$$
$$= \frac{N}{N+J-1}\,\frac{N-1+J}{J} = \frac{N}{J} \qquad (4)$$

as expected, from the network symmetry.

(ii) From (3),
$$\lim_{N\to\infty} TH_j(N) = \lim_{N\to\infty}\frac{N}{N+J-1}\mu = \mu \qquad (5)$$

which is the "bottleneck" proc. rate for this network.

The result of (5) is expected, since $N \to \infty$ implies that the network servers will never starve for work.

(iii) Still we have a very symmetric role for the network workstations, and therefore, the analysis of the previous parts carries over verbatim to this new case.

In fact, the same analysis applies to any other CQN where all workstations have a fully symmetrical role in the network operations. Such another example is the case where all workstation servers have exponential proc. times with the same rate, and the network has a "ring" topology, i.e., customers move deterministically from WS_j to WS_{j+1}, for $j = 1, \dots, J-1$, and from WS_J back to WS_1.

Problem 3:

From the results of the non-preemptive priority queues that were discussed in the lectures, and assuming that class 1 customers have the higher priority, we have:

$$E[W_1^{(n)}] = \frac{R^{(n)}}{1-\rho_1} \; ; \; E[W_2^{(n)}] = \frac{R^{(n)}}{(1-\rho_1)(1-\rho_1-\rho_2)}$$

$$R^{(n)} = \frac{\lambda_1}{2}\left(\sigma_{P_1}^2 + \tau_{P_1}^2\right) + \frac{\lambda_2}{2}\left(\sigma_{P_2}^2 + \tau_{P_2}^2\right)$$

Then, letting $TCR^{(1)}$ denote the expected total charging rate we have:

$$TCR^{(1)} = \sum_{i=1}^{2} \lambda_i \, E[W_i^{(n)}] \cdot C_i$$

since every time unit we get, on average, λ_i customers from class i, and each of these customers will incur an expected gain of $E[W_i^{(n)}] \cdot C_i$. Substituting in the last expression the previous results, we get:

$$TCR^{(1)} = \lambda_1 \frac{R^{(n)}}{1-\rho_1} c_1 + \lambda_2 \frac{R^{(n)}}{(1-\rho_1)(1-\rho_1-\rho_2)} c_2 =$$

$$= R^{(n)} \left[\frac{c_1}{\tau_{\rho_1}} \frac{\rho_1}{1-\rho_1} + \frac{c_2}{\tau_{\rho_2}} \frac{\rho_2}{(1-\rho_1)(1-\rho_1-\rho_2)} \right] =$$

$$= R^{(n)} \left[a_1 \frac{\rho_1}{1-\rho_1} + a_2 \frac{\rho_2}{(1-\rho_1)(1-\rho_1-\rho_2)} \right]$$

If we switch the priorities of the two classes then the corresponding total charging rate will be:

$$TCR^{(2)} = R^{(n)} \left[a_2 \frac{\rho_2}{1-\rho_2} + a_1 \frac{\rho_1}{(1-\rho_2)(1-\rho_1-\rho_2)} \right]$$

We want $TCR^{(1)} > TCR^{(2)}$, which translates to:

$$\frac{a_1\rho_1}{1-\rho_1} + \frac{a_2\rho_2}{(1-\rho_1)(1-\rho_1-\rho_2)} > \frac{a_2\rho_2}{1-\rho_2} + \frac{a_1\rho_1}{(1-\rho_2)(1-\rho_1-\rho_2)}$$

$$\Leftrightarrow \quad a_1\rho_1 \left[\frac{1}{1-\rho_1} - \frac{1}{(1-\rho_2)(1-\rho_1-\rho_2)} \right] >$$

$$a_2\rho_2 \left[\frac{1}{1-\rho_2} - \frac{1}{(1-\rho_1)(1-\rho_1-\rho_2)} \right] \Leftrightarrow$$

$$\Leftrightarrow a_1 \rho_1 \left[(1-\rho_2)(1-\rho_1-\rho_2) - (1-\rho_1)\right] >$$

$$a_2 \rho_2 \left[(1-\rho_1)(1-\rho_1-\rho_2) - (1-\rho_1)\right]$$

The last implication holds because $1-\rho_1-\rho_2 > 0$ for stability.

We also have:

$$(1-\rho_2)(1-\rho_1-\rho_2) - (1-\rho_1) =$$

$$= \cancel{1-\rho_1} - \rho_2 - \rho_2(1-\rho_1-\rho_2) - \cancel{(1-\rho_1)} =$$

$$= -\rho_2 \, (2 - \rho_1 - \rho_2)$$

Similarly

$$(1-\rho_1)(1-\rho_1-\rho_2) - (1-\rho_2) =$$

$$= -\rho_1 \, (2 - \rho_1 - \rho_2)$$

Hence

$$TCR^{(1)} > TCR^{(2)} \Leftrightarrow$$

$$\Leftrightarrow -a_1 \rho_1 \rho_2 (2-\rho_1-\rho_2) > -a_2 \rho_2 \rho_1 (2-\rho_1-\rho_2) \Leftrightarrow$$

$$\Leftrightarrow a_1 < a_2 \Leftrightarrow \frac{c_1}{\tau_{\rho_1}} < \frac{c_2}{\tau_{\rho_2}} \Leftrightarrow \frac{\tau_{\rho_1}}{c_1} > \frac{\tau_{\rho_2}}{c_2}$$

Remark: The proof of the previous result is a typical example of the proofs appearing in the, so called, interchange arguments that establish the optimality of certain ordering rules in scheduling theory.

Problem 4

(i) Stability:

Average lot size $= \frac{1}{4}(2+3+4+5) = 3.5$ parts (1)

Expected lot proc. time $t_b = 3.5 \times 3 = 10.5$ min (2)

Hence,

$$\rho = r_b \cdot t_b = 5\,hr^{-1} \times \frac{10.5}{60}\,hr = \underline{0.875 < 1} \quad (3)$$

MVA

$$TH = 5\,\frac{lots}{hr} \times 3.5\,\frac{parts}{lot} = 17.5 \text{ parts/hr} \quad (4)$$

For the lot waiting time in the queue, we have

$$E[W] \approx \frac{C_{ba}^2 + C_{bp}^2}{2} \frac{\rho}{1-\rho} t_b \quad (5)$$

Also, since lots arrive according to a Poisson process,

$$C_{ba}^2 = 1 \quad (6)$$

Next we compute C_{bp}^2, i.e, the SCV for the batch proc. times.

Letting r.v.'s T_p and T_b denote, respectively the part and the batch proc. times, we have:

$$T_b = \sum_{i=1}^{N} T_p^{(i)} \qquad (7)$$

where r.v. N denotes the number of parts in the batch.

From the theory of compound r.v.'s (c.f. pages 313–314 in the course notes)

$$Var[T_b] = Var[T_p]\,E[N] + E^2[T_p]\,Var[N] \qquad (8)$$

$$E[T_p] = 3 \text{ min} \; ; \; Var[T_p] = 1 \text{ min}^2$$

$$E[N] = 3.5 \quad \text{and} \qquad (9)$$

$$Var[N] = E[N^2] - E^2[N] =$$

$$= \frac{2^2+3^2+4^2+5^2}{4} - 3.5^2 = 1.25 \qquad (10)$$

Hence $Var[T_b] = 1 \cdot 3.5 + 3^2 \cdot 1.25 = 14.75 \text{ min}^2$

Finally,

$$c_{b_p}^2 = SCV[T_b] = \frac{Var[T_b]}{E^2[T_b]} = \frac{14.75}{10.5^2} \simeq 0.134$$

and from (5)

$$E[W] = \frac{1+0.134}{2} \cdot \frac{0.875}{1-0.875} \, 10.5 = 41.6745 \text{ min}$$

Also

$$E[S] = E[W] + t_b = 41.6745 + 10.5 = 52.1745 \text{ min}$$

$$E[X_q] = TH \cdot E[W] = \frac{5}{60} \times 41.6745 \simeq 3.47 \text{ lots}$$

$$\simeq 3.47 \times 3.5 = 12.145 \text{ parts}$$

$$E[X] = TH \cdot E[S] = \frac{5}{60} \cdot 52.1745 \simeq 4.35 \text{ lots}$$

$$= 4.35 \times 3.5 = 15.22 \text{ parts.}$$

(ii) We have

$$f_p(r_p) = 2 + \frac{r_p - 15}{25-15} \cdot 3 = 0.3\, r_p - 2.5 \qquad (11)$$

For stability, we need:

$$5 \times \left(3.5 \times \frac{1}{r_p}\right) < 1 \Rightarrow$$

$$\Rightarrow \quad r_p > 5 \times 3.5 = 17.5 \text{ parts/hr}$$

Now

$$t_b = E[N] \cdot t_p = E[N]/r_p = \frac{3.5}{r_p} \text{ hr} =$$

$$= \frac{3.5 \times 60}{r_p} \text{ min} = \frac{210}{r_p} \text{ min}$$

Also, from (8) and part (i):

$$Var[T_b] = 1 \cdot 3.5 + 1.25 \cdot \left(\frac{60}{r_p}\right)^2 = 3.5 + \frac{4500}{r_p^2}$$

and

$$scv[T_b] = \frac{var[T_b]}{E^2[T_b]} = \frac{3.5 + \frac{4500}{r_p^2}}{(210/r_p)^2} =$$

$$\simeq 8 \times 10^{-5} r_p^2 + 0.102$$

Then,

$$\rho = 5 \times \frac{3.5}{r_p} = \frac{17.5}{r_p}$$

and

$$E[S; r_p] = \frac{1 + 0.102 + 8\times 10^{-5} r_p^2}{2} \frac{17.5/r_p}{1 - 17.5/r_p} \frac{210}{r_p} + \frac{210}{r_p}$$

$$= \frac{105}{r_p^2 - 17.5 r_p} \left[0.0014 r_p^2 + 2 r_p - 15.75 \right] \qquad (12)$$

The expected total cost per part when operating at rate r_p is $0.05\, E[S; r_p] + f_p(r_p)$.

Hence, to get an optimal value for r_p, we must solve:

$$\left.\begin{array}{c} \min\ 0.05\, E[S; r_p] + f_p(r_p) \\ \text{s.t.} \\ 17.5 < r_p \leq 25 \end{array}\right\} \qquad (13)$$

where $E[S; r_p]$ and $f_p(r_p)$ are obtained from (12) and (11).

Searching over the interval [17.5, 25] with the indicated resolution of 0.1, we can see that $\underline{r_p^* = 22}$.

Problem 5

(i) Stability:

The batch arrival rates in the waiting queue are

$$\lambda_b(1) = \frac{\lambda(1)}{5} = \frac{5}{5} = 1$$

$$\lambda_b(2) = \frac{4}{5} \quad 0.8$$

Hence, the arriving workload per time unit is

$$\lambda_b(1) \cdot t_p(1) + \lambda_b(2)\, t_p(2) =$$

$$= 1 \cdot 0.5 + 0.8 \cdot 0.35 = \underline{0.78 < 1}.$$

MVA:

Since the workstation is stable,

$$TH(1) = \lambda(1) \quad \text{and} \quad TH(2) = \lambda(2).$$

Also

$$u(1) = 1 \cdot 0.5 = 0.5$$

$$u(2) = 0.8 \cdot 0.35 = 0.28$$

and $u = u(1) + u(2) = 0.78$

The average times for the formation of batch from each part type are respectively:

$$WTBT(1) = \frac{5-1}{2} \cdot \frac{1}{\lambda(1)} = \frac{2}{5} = 0.4$$

$$WTBT(2) = \frac{5-1}{2} \frac{1}{\lambda(2)} = \frac{2}{4} = 0.5$$

In order to compute $E[W]$ for the formed batches, we model the considered workstation as a $G/G/1$ queue modeling a single job type with the following parameters:

Arrival rate $\lambda = \lambda(1)/k + \lambda(2)/k = \frac{5}{5} + \frac{4}{5} = 1.8$

SCV for the inter-arrival times $c_a^2 =$

$$\frac{\lambda(1)/k}{\lambda} c_{a_b}^2(1) + \frac{\lambda(2)/k}{\lambda} c_{a_b}^2(2) =$$

$$= \frac{1}{k\lambda} \left[\lambda(1) \frac{c_a^2(1)}{k} + \lambda(2) \frac{c_a^2(2)}{k} \right] =$$

$$= \frac{1}{k^2 \lambda} \left[\lambda(1) c_a^2(1) + \lambda(2) c_a^2(2) \right] =$$

$$= \frac{1}{5^2 \, 1.8} \left[5 \times 1.5 + 4 \times 2.2 \right] = 0.362$$

The proc. times for this queue follow the distribution

$$\begin{cases} 0.5 & \text{w.p. } \frac{\lambda(1)/k}{\lambda} = \frac{5/5}{1.8} \simeq 0.556 \\ 0.35 & \text{w.p. } (\lambda(2)/k)/\lambda \simeq 0.444 \end{cases}$$

Hence,

$$t_p = 0.5 \times 0.556 + 0.35 \times 0.444 = 0.4334$$

$$\sigma_p^2 = (0.5 - 0.4334)^2 \times 0.556 +$$
$$+ (0.35 - 0.4334)^2 \times 0.35 = 0.0049$$

and

$$c_p^2 = \frac{\sigma_p^2}{t_p^2} = \frac{0.0049}{0.4334^2} \simeq 0.026$$

So

$$E[W] = \frac{c_a^2 + c_p^2}{2} \frac{u}{1-u} t_p =$$

$$= \frac{0.362 + 0.026}{2} \frac{0.78}{1 - 0.78} 0.4334 = 0.298$$

Furthermore,

$$E[S](1) = WTBT(1) + E[W] + t_p(1) =$$
$$= 0.4 + 0.298 + 0.5 = 1.198$$

$$E[S](2) = WTBT(2) + E[W] + t_p(2) =$$
$$= 0.5 + 0.298 + 0.35 = 1.148$$

and from Little's law,

$$E[X](1) = TH(1)\, E[S](1) = 5 \times 1.198 = 5.99$$

$$E[X](2) = TH(2)\, E[S](2) = 4 \times 1.148 = 4.592$$

We can also compute the expected number of parts in the waiting queue from each part type using Little's law on the waiting queue:

$$E[X_q](1) = TH(1) \cdot E[W] =$$
$$= 5 \times 0.298 = 1.49$$
$$E[X_q](2) = TH(2)\, E[W] =$$
$$= 4 \times 0.298 = 1{,}192$$

(ii) First notice that, for stability, we need:

$$u = \frac{5}{K} 0.5 + \frac{4}{K} 0.35 < 1 \Rightarrow K > 3.9 \Rightarrow$$
$$\Rightarrow \underline{K \geq 4} \quad (1)$$

Also, working as in part (i), we have

$$E[W] = \frac{c_a^2 + c_p^2}{2} \frac{u}{1-u} t_p$$

where $t_p = 0.4334$, $c_p^2 = 0.026$

$u = 3.9/K$ (from (1))

and

$$c_a^2 = \frac{\lambda(1)}{\lambda(1)+\lambda(2)} \frac{c_a^2(1)}{K} + \frac{\lambda(2)}{\lambda(1)+\lambda(2)} \frac{c_a^2(2)}{K} =$$
$$= \frac{1}{K}\left[\frac{5}{9} \cdot 1.5 + \frac{4}{9} 2.2\right] = \frac{1.811}{K}$$

Then,

$$E[W] = \frac{(1.811/K) + 0.026}{2} \frac{3.9/K}{1 - 3.9/K} \cdot 0.4334$$

$$= \frac{1.811 + 0.026K}{K} \frac{3.9}{K - 3.9} 0.2167 = \frac{0.022K + 1.53}{K(K-3.9)}$$

The batch arrival rate at the waiting queue is

$$r_b = \frac{\lambda(1)}{K} + \frac{\lambda(2)}{K} = \frac{5}{K} + \frac{4}{K} = \frac{9}{K}$$

and from Little's law

$$E[B_q] = r_b \cdot E[W] = \frac{9}{K} \frac{0.022K + 1.53}{K(K-3.9)} =$$

$$= \frac{0.198K + 13.77}{K^2(K-3.9)}$$

So, we need to solve

$$\begin{cases} \min \dfrac{0.198K + 13.77}{K^2(K-3.9)} \\ \text{s.t.} \\ K \geq 4 \\ K \in \mathbb{Z}^+ \end{cases}$$

But it is clear that

$$\lim_{K \to \infty} \frac{0.198K + 13.77}{K(K-3.9)} = \emptyset$$

and $K^* \to \infty$.

This result is explained by the fact that from the view point of the considered G/G/1 queue, larger values of K imply lower batch arrival rates. In particular, as $K \to \infty$, these arrival rates, and the server utilization, are driven to zero. But then, E[W] and E[B_q] will go to zero, as well.

Of course $K \to \infty$ implies that WIBT $\to \infty$. Hence, parts will be stuck in this part of the overall operation.

At a higher level, this example intends to show that the optimization of the considered systems must be based on a holistic view of their operation; otherwise, there might be unintended consequences!