ISYE 7201: Production & Service Systems
Spring 2020
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2nd Midterm Exam (Take Home)
Release Date: March 4, 2020
Due Date: March 14, 2020

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

You can send me your responses as a pdf file attached to an email. This pdf file can be a scan of a hand-written document, but, please, write your answers very clearly and thoroughly. Also, report any external sources (other than your textbook) that you referred to while preparing the solutions.

Problem 1 (20 points): In class we showed that a counting process $\{N(t), t \geq 0\}$ where the inter-event times are independent, exponentially distributed random variables with a common rate λ, is Poisson with the same rate. However, during the proof of this result, I skipped the part that would establish that the considered process $N(t)$ has independent increments. Provide the missing argument.

Problem 2 (20 points): Consider an elevator that starts in the basement and travels upwards. Let N_i denote the number of people that get in the elevator at floor i. Assume that N_i are independent and that N_i is Poisson distributed with mean λ_i. Each person entering at floor i will, independent of everything else, get off at floor j with probability p_{ij}. Furthermore, $\sum_{j>i} p_{ij} = 1.0$. Finally, let O_j denote the number of people getting off the elevator at floor j.

i. **(5 pts)** Compute $E[O_j]$.

ii. **(10 pts)** What is the distribution of O_j?

iii. **(5 pts)** What is the joint distribution of O_j and O_k?

Please, provide complete justifications for your answers.

Problem 3 (20 points): Consider the motion of three indistinguishable balls on a linear array of positions indexed by the positive integers, such that one or more balls can occupy the same position. Suppose that at time $t = 0$ there is one ball at position one, one ball at position 2 and one ball at position three. Given the positions of the balls at some integer time t, their positions at time $t + 1$ are determined as follows: One of the balls in the leftmost occupied position is picked up, and one of the other two balls is selected at random (but not moved) with each choice having probability one half. The ball that was picked up is then placed one position to the right of the selected ball.

i. **(5 pts)** Define a three-state Markov process that tracks the relative positioning of the balls at each discrete time $t \in Z_0^+$. Describe the meaning of each state, and give the one-step transition probability matrix for this process. (*Hint:* Exploit the fact that the balls are indistinguishable, and don't include the actual positions occupied by the balls in your definition of the process state.)

ii. **(5 pts)** Find the equilibrium distribution of the process defined in part (i).

iii. **(5 pts)** As time progresses, the entire set of balls moves to the right, and the average speed for this motion has a limiting value with probability one. Find this limiting value (*Hint:* Consider the ball motions in each state of the discrete-time Markov chain that you defined in the previous parts of this problem, and induce a notion of "speed" from these motions.)

iv. **(5 pts)** Consider the following continuous-time version of the above problem: Given the current state at time t, a move as described in the opening part of this problem, happens in the interval $[t, t+h]$ with probability $h + o(h)$. Provide the infinitesimal generator matrix Q for the corresponding CTMC, compute the equilibrium distribution for this process, and identify the long-term average speed of the ball drifting in this new regime.

Problem 4 (20 points) In each of the following four cases, compute

$$\lim_{t\to\infty} P(X_t = 2|X_0 = 1)$$

for the Markov chain $(X_t)_{t\geq 0}$ with the given infinitesimal generator matrix on $\{1, 2, 3, 4\}$:

a. (5 pts)

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

b. (5 pts)

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

c. (5 pts)

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

d. (5 pts)

$$\begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In the above representation, states that have their rows in the infinitesimal generator matrix equal to the **0** vector, are absorbing states for the corresponding process.

Problem 5 (20 points): When I introduced the concept of the CTMC, I discussed the modeling of an M/M/1 queueing station as an example of such a process. Now consider another single-server queueing station where customers arrive in a Poisson stream of rate λ. Each customer has a service requirement distributed according to an Erlang(2,μ) distribution. Service times are independent from each other, and of the arrival process. Also, customers joining the queue of this station are processed First-Come-First-Serve, and the server operates in a non-failing and non-idling mode.

i. **(10 pts)** Model the operation of this queueing station as a CTMC, explaining clearly the meaning of each state in your model, and detailing all the parameters that define the transitional dynamics of this CTMC.

ii. **(10 pts)** Also, show that this CTMC will have a limiting distribution if and only if $\lambda/\mu < 1/2$.

ISYE 7201 - SPRING 2020

MIDTERM II SOLUTIONS

Problem 1:

One way to prove this result is as follows: Consider the time points t_1, t_2, t_3 and t_4 with $t_1 < t_2 < t_3 < t_4$. Then, the intervals (t_1, t_2) and (t_3, t_4) are non-overlapping, and we have:

$$P[N(t_4) - N(t_3) = x \mid N(t_2) - N(t_1) = y] =$$

$$= \sum_{z=0}^{\infty} P[N(t_4) - N(t_3) = x \mid N(t_3) - N(t_2) = z \wedge N(t_2) - N(t_1) = y] \cdot$$

$$\cdot P[N(t_3) - N(t_2) = z \mid N(t_2) - N(t_1) = y] =$$

$$= \sum_{z=0}^{\infty} \frac{P[N(t_4) - N(t_3) = x \wedge N(t_3) - N(t_2) = z \wedge N(t_2) - N(t_1) = y]}{\cancel{P[N(t_3) - N(t_2) = z \wedge N(t_2) - N(t_1) = y]}} \cdot$$

$$\cdot \frac{\cancel{P[N(t_3) - N(t_2) = z \wedge N(t_2) - N(t_1) = y]}}{P[N(t_2) - N(t_1) = y]} =$$ (from memoryless property of the exp. distr.)

$$= \sum_{z=0}^{\infty} \frac{P[N(t_4) - N(t_3) = x] \cdot P[N(t_3) - N(t_2) = z] \cdot \cancel{P[N(t_2) - N(t_1) = y]}}{\cancel{P[N(t_2) - N(t_1) = y]}} =$$

$$= P[N(t_4) - N(t_3) = x] \cdot \sum_{z=0}^{\infty} P[N(t_3) - N(t_2) = z] =$$

$$= P[N(t_4) - N(t_3) = x]$$

Problem 2:

Let O_{ij} denote the number of people who got in the elevator at floor i and got off at floor j. Then,

$$P[O_{ij}=n] = \sum_{k=0}^{\infty} P[O_{ij}=n \mid N_i=k]\, P[N_i=k] =$$

$$= \sum_{k=n}^{\infty} \binom{k}{n} p_{ij}^{n} (1-p_{ij})^{k-n} \frac{e^{-\lambda_i}\lambda_i^{k}}{k!} =$$

$$= \sum_{k=n}^{\infty} \frac{\cancel{k!}}{n!\,(k-n)!} p_{ij}^{n} (1-p_{ij})^{k-n} \frac{e^{-\lambda_i}\lambda_i^{k}}{\cancel{k!}} =$$

$$= \sum_{k=n}^{\infty} \left[\frac{(\lambda_i p_{ij})^n}{n!} e^{-\lambda_i p_{ij}}\right]\left[\frac{(\lambda_i(1-p_{ij}))^{k-n}}{(k-n)!} e^{-\lambda_i(1-p_{ij})}\right] =$$

$$= \frac{(\lambda_i p_{ij})^n}{n!} e^{-\lambda_i p_{ij}} \sum_{u=0}^{\infty} \frac{(\lambda_i(1-p_{ij}))^u}{u!} e^{-\lambda_i(1-p_{ij})} = (*)$$

$$= \frac{(\lambda_i p_{ij})^n}{n!} e^{-\lambda_i p_{ij}} \Rightarrow O_{ij} \sim \text{Poisson}(\lambda_i p_{ij})$$

Furthermore, $O_j = \sum_{i<j} O_{ij}$ where the O_{ij} involved are Poisson distributed and mutually independent. In addition, the moment generating function of each O_{ij} is

$$MGF(O_{ij}) = E[\exp(O_{ij}t)] = e^{\lambda_i p_{ij}(e^t-1)}$$

where the last part results from the fact that $O_{ij} \sim \text{Poisson}(\lambda_i p_{ij})$

Also,

$$MGF(O_j) = E[\exp(O_j t)] = E\left[\exp\left(t\sum_{i<j} O_{ij}\right)\right] \overset{\text{from independence of } O_{ij}}{=}$$

$$= \prod_{i<j} E[\exp(O_{ij}t)] = \cancel{E\left[\exp\left(t\sum_{i<j} O_{ij}\right)\right]} \prod_{i<j} e^{\lambda_i p_{ij}(e^t-1)}$$

$$= e^{\left(\sum_{i<j} \lambda_i P_{ij}\right)(e^t - 1)} \implies O_j \sim \text{Poisson}\left(\sum_{i<j} \lambda_i P_{ij}\right)$$

Furthermore, the above result implies that $E[O_j] = \sum_{i<j} \lambda_i P_{ij}$.

Finally, the computation of the pmf of O_{ij} implies that this r.v. is independent from the number of passengers that enter the elevator at floor i and get off at some floor other than j (see the part highlighted by (*) in the corresponding computation above).

This result further implies that O_j and O_k are independent r.v.'s for $j \neq k$. But then

$$P(O_j = n \wedge O_k = m) = P(O_j = n) \cdot P(O_k = m) =$$

$$= \left[e^{-\sum_{i<j} \lambda_i P_{ij}} \frac{\left(\sum_{i<j} \lambda_i P_{ij}\right)^n}{n!} \right] \left[e^{-\sum_{i<k} \lambda_i P_{ik}} \frac{\left(\sum_{i<k} \lambda_i P_{ik}\right)^m}{m!} \right]$$

In addition, a computation similar to that provided above for O_j, will establish that $O_j + O_k \sim \text{Poisson}\left(\sum_{i<j} \lambda_i P_{ij} + \sum_{i<k} \lambda_i P_{ik}\right)$

Problem 3:

(i) With the initial positioning of the balls and the dynamics of their motion that are provided in the problem description, it is not hard to see that at any time point the relative positioning of the balls will be described by the following three states:

(a) 111: the three balls occupy three consecutive positions.
(b) 12: one ball is in the leftmost occupied position and the other two balls are one position to the right of it.
21: two balls are in the leftmost occupied position and one ball in the position to the right of them.

Furthermore, this state evolves according to the following one-step transition prob. matrix:

$$P = \begin{array}{c} \\ 111 \\ 12 \\ 21 \end{array} \begin{array}{c} \begin{array}{ccc} 111 & 12 & 21 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix} \end{array}$$

(ii) The DTMC defined in step (i) is finite-state and irreducible; therefore it has an equilibrium distribution that is computed as follows:

$$\underline{\Pi} = (\Pi_{111}\ \Pi_{12}\ \Pi_{21}) = (\Pi_{111}\ \Pi_{12}\ \Pi_{21}) \cdot P \quad ; \quad \sum_{x \in \{111, 12, 21\}} \Pi_x = 1$$

The solution of the above system of equations will result in

$$\underline{\Pi} = (1/3,\ 1/3,\ 1/3).$$

It can also be checked that the considered MC is aperiodic (notice, for instance, the self-loop at state 111) and therefore, the ~~considered~~ computed distribution also constitutes the limiting distr. for this MC.

(iii) After each visit at the states 111 and 12, the leftmost position of the ball configuration advances one position to the right. On the other hand, a visit to state 21 does not advance the leftmost position of the ball configuration. Thus the average speed of the balls' drift to the right is:

$$2/3 \cdot 1 + 1/3 \cdot 0 = 2/3 \text{ . slots per time period.}$$

(iv) This part of the problem description implies that the ball-advancing events follow a Poisson distribution with rate $\lambda = 1.0$. Using the same state space as in the previous parts, we can model the dynamics of the ball motion as a CTMC with infinitesimal generator matrix

$$Q = \begin{array}{c|ccc} & 111 & 12 & 21 \\ \hline 111 & -0.5 & 0.5 & 0 \\ 12 & 0 & -1 & 1 \\ 21 & 0.5 & 0.5 & -1 \end{array}$$

The limiting distribution for this CTMC is obtained by

$$\underbrace{(p_{111}\ p_{12}\ p_{21})}_{\underline{p}} Q = \emptyset \ ; \qquad \underline{p} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = 1$$

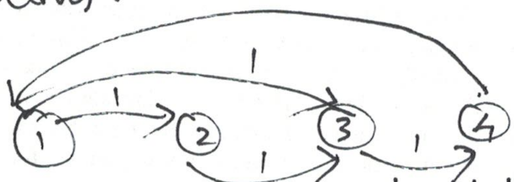
The solution of the above system is still $\underline{p} = (1/3, 1/3, 1/3)$

(<u>Remark:</u> This should be expected since all three states of this CTMC have a <u>uniform</u> departure rate $\lambda = 1$; then the above result is a demonstration of the corresponding result that we established for uniformized CTMCs).

Finally, the average ~~drift~~ speed for the ball drift can be computed equal to $2/3$ slots per time unit, in a way similar to that followed in part (iii).

Problem 4:

a) The state transition diagram (STD) for the CTMC is as follows:



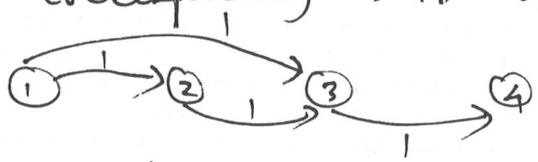
It is clear that the embedded DTMC of this process is irreducible and positive recurrent (since it is also finite state). Therefore the CTMC has a limiting distribution obtained as follows:

$$(p_1 \;\; p_2 \;\; p_3 \;\; p_4)\begin{bmatrix} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}; \quad \sum_{i=1}^{4} p_i = 1 \rightarrow$$

$$\cdots \Rightarrow (p_1 \;\; p_2 \;\; p_3 \;\; p_4) = (1/6 \;\; 1/6 \;\; 1/3 \;\; 1/3)$$

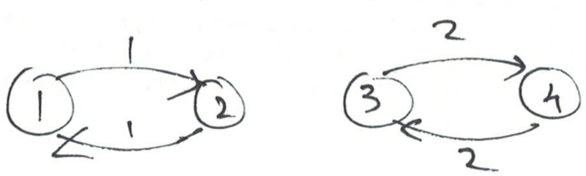
Hence $\lim_{t\to\infty} P(X_t = 2 \mid X_0 = 1) = p_2 = 1/6$.

b) The corresponding STD is



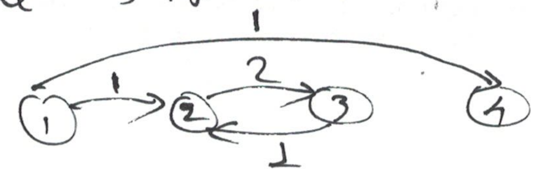
Hence state 4 is the unique absorbing state for this process and the sought probability is equal to 0.

c) In this case, the STD is:



Hence, if the process is started at state $X_0 = 1$, it is trapped in the subspace that is defined by states 1 and 2. The underlying DTMC is irreducible and therefore the considered continuous-time MC over these two states has a limiting distribution. Furthermore, from symmetry, it can be seen that the corresponding state probabilities are $1/2$. This is also the value for $\lim_{t \to \infty} P(X_t = 2 \mid X_0 = 1)$.

d) The STD is:



Starting from state 1, the considered process will be absorbed either in state 4 or the irreducible subspace defined by states 2 and 3. Furthermore, it is easy to see that the corresponding absorption probabilities are equal to $1/2$.

In the subspace of states 2 and 3, the process has the limiting distribution $(P_2 \ P_3)\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} = 0$; $P_2 + P_3 = 1 \Rightarrow$

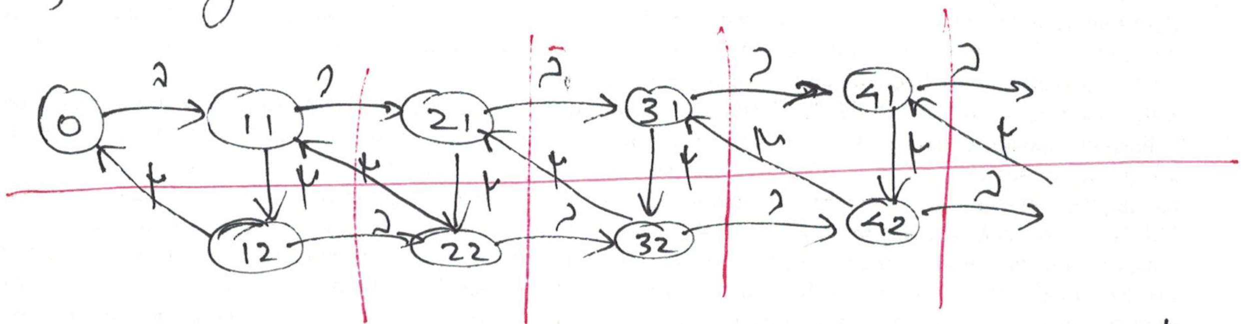
$\Rightarrow \quad (P_2 \ P_3) = (1/3, 2/3)$

Finally $\lim_{t \to \infty} P(X_t = 2 \mid X_0 = 1) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

Problem 5:

8

(i) The CTMC modeling the dynamics of the considered queueing station can be represented completely by the following STD:



The states (ij) in the above STD with $i = 1, 2, \ldots$ and $j \in \{1, 2\}$ should be understood as follows:

- i reports the # of customers in the station.
- j reports the stage of the customer in service.

(ii) First, we show the necessity of the condition $\lambda/\mu < 1/2$ for the existence of a limiting distribution for this CTMC. Consider the horizontal "cut" and the vertical "cuts" annotated by red lines in the above STD. If there exists a limiting distribution, then the flow between the two parts of the STD that are defined by each cut must be equal in both directions. Hence, from the horizontal cut we get:

$$\mu \sum_{i=1}^{\infty} \pi_{i1} = \mu \sum_{i=1}^{\infty} \pi_{i2} \Rightarrow \sum_{i=1}^{\infty} \pi_{i1} = \sum_{i=1}^{\infty} \pi_{i2} \qquad (1)$$

Also, from the vertical cuts we have:

$$\forall i \geq 1, \quad (\pi_{i1} + \pi_{i2})\lambda = \pi_{(i+1)2}\,\mu \qquad (2)$$

Summing the last set of equations over all i, we get:

$$\lambda \sum_{i=1}^{\infty} (\pi_{i1} + \pi_{i2}) = \mu \sum_{i=1}^{\infty} \pi_{(i+1)2} \qquad (3)$$

From (1) and (3):

$$2\lambda \sum_{i=1}^{\infty} \pi_{i2} = \mu \sum_{i=1}^{\infty} \pi_{(i+1)2} = \mu \left[\sum_{i=1}^{\infty} \pi_{i2} - \pi_{i2} \right] < \mu \sum_{i=1}^{\infty} \pi_{i2}$$

$$\Rightarrow \quad 2\lambda < \mu \quad \Rightarrow \quad \lambda/\mu < 1/2$$

In the above derivation, we have used the fact that $0 < \sum_{i=1}^{\infty} \pi_{i2} < \infty$ if there exists a limiting distribution.

(Remark: Also, let me add that in the above discussion π denotes the limiting distribution for the CTMC, not the embedded DTMC.)

In order to show the sufficiency of the considered condition $\lambda/\mu < 1/2$ for the existence of the limiting distribution, we shall show that under this condition, Equations (1) and (2) above define completely the sought distribution. More specifically, as shown above, Eqs (1) and (2) further imply that

$$\sum_{i=1}^{\infty} (2\lambda \pi_{i2} - \mu \pi_{(i+1)2}) = 0 \qquad (4)$$

This equation can be satisfied by setting

$$\forall i \geq 1, \quad 2\lambda \pi_{i2} = \mu \pi_{(i+1)2} \Rightarrow \pi_{(i+1)2} = \frac{2\lambda}{\mu} \pi_{i2} \quad (5)$$

Equation (5) further implies that

$$\forall i \geq 1, \quad \pi_{i2} = \left(\frac{2\lambda}{\mu}\right)^{i-1} \pi_{12} \quad (6)$$

Hence,

$$\sum_{i=1}^{\infty} \pi_{i2} = \sum_{i=1}^{\infty} \left(\frac{2\lambda}{\mu}\right)^{i-1} \pi_{12} = \pi_{12} \sum_{i=1}^{\infty} \left(\frac{2\lambda}{\mu}\right)^{i} =$$

$$= \pi_{12} \frac{1}{1 - 2\lambda/\mu} = \pi_{12} \frac{\mu}{\mu - 2\lambda} \quad (7)$$

In the above computation we have used the working assumption $\lambda/\mu < 1/2 \Rightarrow \frac{2\lambda}{\mu} < 1$, and the results for the convergence of a geometric series.

Eqs (1) and (7) subsequently imply that:

$$1 = \pi_0 + \sum_{i=1}^{\infty} \pi_{i1} + \sum_{i=1}^{\infty} \pi_{i2} = \pi_0 + 2 \sum_{i=1}^{\infty} \pi_{i2} = \pi_0 + \frac{2\mu}{\mu - 2\lambda} \pi_{12} \quad (8)$$

Furthermore, from the flow-balance equation at state $\emptyset$, we have:

$$\pi_0 \lambda = \pi_{12} \mu \Rightarrow \pi_{12} = \lambda/\mu \, \pi_0 \quad (9)$$

From (8) and (9):

$$\pi_0 \left(1 + \frac{2\cancel{\mu}}{\mu - 2\lambda} \frac{\lambda}{\cancel{\mu}}\right) = 1 \Rightarrow \pi_0 = \frac{\mu - 2\lambda}{\mu - 2\lambda + 2\lambda} = 1 - \frac{2\lambda}{\mu} \quad (10)$$

Then, Eqs (6), (9) and (10) also imply that:

$$\forall i \geq 1 \quad \pi_{i2} = \left(\frac{2\lambda}{\mu}\right)^{i-1} \frac{\lambda}{\mu} \left(1 - \frac{2\lambda}{\mu}\right) \quad (11)$$

The computation of the π_{i1}, $i=1,2,\dots$ can be performed from the already obtained results using the flow balance equation for each state $i1$, $i=1,2,\dots$

Hence, for $i=1$, we have

$$(\lambda+\mu)\pi_{11} = \lambda\pi_0 + \mu\pi_{22} \Rightarrow \pi_{11} = \frac{\lambda}{\lambda+\mu}\pi_0 + \frac{\mu}{\lambda+\mu}\pi_{22}$$

for $i=2$,

$$(\lambda+\mu)\pi_{21} = \lambda\pi_{11} + \mu\pi_{32} \Rightarrow \pi_{21} = \frac{\lambda}{\lambda+\mu}\pi_{11} + \frac{\mu}{\lambda+\mu}\pi_{32}$$

More generally, for $i\geq 2$:

$$(\lambda+\mu)\pi_{i1} = \lambda\pi_{(i-1)1} + \mu\pi_{(i+1)2} \Rightarrow \pi_{i1} = \frac{\lambda}{\lambda+\mu}\pi_{(i-1)1} + \frac{\mu}{\lambda+\mu}\pi_{(i+1)2}$$

It is interesting to notice that each π_{i1}, $i=1,2,\dots$ is obtained as the weighted sum of two already computed probabilities and therefore it does belong in the interval $(0,1)$.

Also, for verification purposes, notice that the summation of the above equations gives

$$(\lambda+\mu)\sum_{i=1}^{\infty}\pi_{i1} = \lambda\pi_0 + \lambda\sum_{i=1}^{\infty}\pi_{i1} + \mu\sum_{i=2}^{\infty}\pi_{i2} \Rightarrow$$

$$\Rightarrow \mu\sum_{i=1}^{\infty}\pi_{i1} = \lambda\pi_0 + \mu\sum_{i=2}^{\infty}\pi_{i2} \overset{(9)}{=} \lambda\frac{\mu}{\lambda}\pi_{12} + \mu\sum_{i=2}^{\infty}\pi_{i2} =$$

$$= \mu\sum_{i=1}^{\infty}\pi_{i2},$$ which is consistent with (1).

Finally, also notice that $2\lambda/\mu = \lambda(2\tau)$ is the average workload arriving at this station per time unit. And since the station has a single nonfaulting and nonidling server, it is reasonable to expect that the stability condition for this station is

$$2\lambda/\mu < 1 \iff \lambda/\mu < 1/2 .$$

This insight also explains the result $\Pi_0 = 1 - 2\lambda/\mu$, since state $\emptyset$ is the only state in which the server is idle.