MIDTERM II SOLUTIONS

Problem 1:

a) First we show that $Y_n = X_n^2$ is not a Markov chain by means of a counter-example. So, let the state space of X be $S = \{-1, 0, 1, 2\}$ and its one-step transition probability matrix be

$$P_X = \begin{array}{c|c} & \begin{array}{cccc} -1 & 0 & 1 & 2 \end{array} \\ \hline \begin{array}{c} -1 \\ 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.25 & 0 & 0.25 & 0.5 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Then, $P[Y_{n+1} = 0 \mid Y_{n-1} = 4, Y_n = 1] = P[X_{n+1} = 0 \mid X_{n-1} = 2, X_n = 1] =$

$$= P[X_{n+1} = 0 \mid X_n = 1] = 0.5$$

The first equality above results from the fact that the considered behavior of process $\{Y_n\}$ can be generated only by the sample path of $\{X_n\}$ that is indicated in the rhs of this equality. The second equality results from the Markov property of $\{X_n\}$.

Next, we consider the following conditional probability for $\{Y_n\}$:

$$P[Y_{n+1} = 0 \mid Y_n = 1] = P[X_{n+1} = 0 \mid X_n = 1 \vee X_n = -1] =$$

$$= \frac{P[X_{n+1} = 0 \wedge (X_n = 1 \vee X_n = -1)]}{P[X_n = 1 \vee X_n = -1]} =$$

$$= \frac{P[(X_{n+1} = 0 \wedge X_n = 1) \vee (X_{n+1} = 0 \wedge X_n = -1)]}{P[X_n = 1 \vee X_n = -1]} =$$

$$= \frac{P[X_{n+1} = 0 \wedge X_n = 1] + P[X_{n+1} = 0 \wedge X_n = -1]}{P[X_n = 1] + P[X_n = -1]} =$$

$$= \underbrace{P[X_{n+1} = 0 \mid X_n = 1]}_{0.5} \frac{P[X_n = 1]}{P[X_n = 1] + P[X_n = -1]} +$$

$$+ \underbrace{P[X_{n+1} = 0 \mid X_n = -1]}_{1 = 0.5 + 0.5} \frac{P[X_n = -1]}{P[X_n = 1] + P[X_n = -1]} = 0.5 + 0.5 \frac{P[X_n = -1]}{P[X_n = 1] + P[X_n = -1]}$$

It can be easily checked that the considered MC $\{X_n\}$ is irreducible, positive recurrent and aperiodic. Hence, it has a limiting distribution with a positive probability for all four states, and therefore, for sufficiently large n,

$$\underbrace{0.5 + 0.5 \frac{P[X_n=-1]}{P[X_n=1]+P[X_n=-1]}}_{P[Y_{n+1}=0 \mid Y_n=1]} > \underbrace{0.5}_{P[Y_{n+1}=0 \mid Y_{n-1}=4,\ Y_n=1]}$$

The above inequality implies that $\{Y_n\}$ is not Markov.

b) On the other hand, for $W_n = X_n^3$, we have:

$$P[W_{n+1}=i_{n+1} \mid W_0=i_0, W_1=i_1, \dots, W_n=i_n] =$$

$$= P[X_{n+1}^3=i_{n+1} \mid X_0^3=i_0, X_1^3=i_1, \dots, X_n^3=i_n] =$$

$$= P[X_{n+1}=\sqrt[3]{i_{n+1}} \mid X_0=\sqrt[3]{i_0}, X_1=\sqrt[3]{i_1}, \dots, X_n=\sqrt[3]{i_n}] =$$

$$= P[X_{n+1}=\sqrt[3]{i_{n+1}} \mid X_n=\sqrt[3]{i_n}] \quad (\text{since } \{X_n\} \text{ is Markov}) =$$

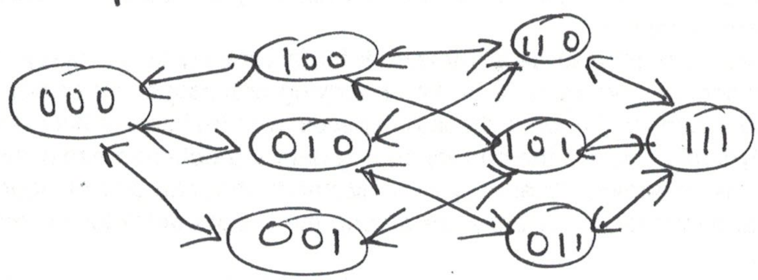
$$= P[X_{n+1}^3=i_{n+1} \mid X_n^3=i_n] =$$

$$= P[W_{n+1}=i_{n+1} \mid W_n=i_n]$$

So $\{W_n\}$ is Markov.

Problem 2:

(i) A compact way to represent this STD is as follows:



All transitions depicted in the above diagram are bidirectional and each single transition occurs with prob. $1/3$.

(ii) The state space of $\{Y_t\}$ is $S_Y = \{0, 1, 2, 3\}$ and the correspondence between the elements of S_Y with the elements of S_X (i.e., the state space of the original process) is as follows:

0 → {(000)}
1 → {(100), (010), (001)}
2 → {(110), (101), (011)}
3 → {(111)}

The above mapping together with the STD developed in part (i) further imply the following facts:

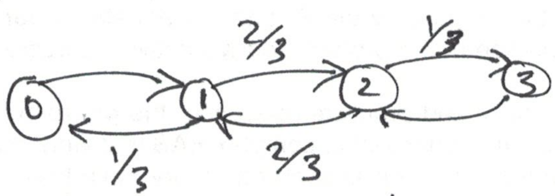
a) When at state 0, process Y_t will transition to state 1 with prob. 1 at the next period.

b) When process Y_t is at state 1, the original process can be at any of the three states (100), (010) and (001). But no matter which is this state, at the next period this process will be at a state mapping at state 2 of Y_t with prob. $2/3$, or at state (000) (i.e., state $\emptyset$ of Y_t) with the remaining probability

c) Similarly, when Y_t is at state 2, the original process will be at a state corresponding to state 1 of Y_t with prob. $2/3$, and to state (111) (i.e., state 3 of Y_t) with prob. $1/3$.

d) Finally, whenever Y_t is at state 3, will be at state 2 in the next period.

Collectively, the above remarks imply the following STD for $\{Y_t\}$:



which implies Markovian behavior.

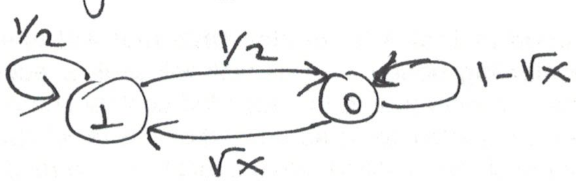
(iii) Since (a) the original process is in state (000) if and only if the induced process $\{Y_t\}$ is at state 0, and (b) every transition in one process corresponds to some transition in the other one, we can get the mean recurrence time of state (000) in the original process by computing the mean recurrence time of state 0 in process $\{Y_t\}$. To get this last result, let a_i, $i=1,2,3$, be the mean time for Y_t to reach state $\emptyset$ starting from state i. Conditioning on the first step out of each of these three states, we have the following equations:

$$\left\{\begin{array}{l} a_1 = 1 + \frac{2}{3}a_2 \\ a_2 = 1 + \frac{2}{3}a_1 + \frac{1}{3}a_3 \\ a_3 = 1 + a_2 \end{array}\right. \Longrightarrow \left\{\begin{array}{l} a_1 = 7 \\ a_2 = 9 \\ a_3 = 10 \end{array}\right.$$

The sought mean recurrence time is $1 + a_1 = 8$.

Problem 3:

The dynamics of this stochastic process can be represented by the following State Transition Diagram (STD):



In the above STD, state 1 implies that the process is in control, and state 0 implies that the process is out of control. Clearly, the DTMC corresponding to the above STD has a limiting distribution (π_0, π_1) that can be computed as follows:

$$(\pi_0 \ \pi_1) = (\pi_0 \ \pi_1)\begin{bmatrix} 1-\sqrt{x} & \sqrt{x} \\ 1/2 & 1/2 \end{bmatrix} \quad ; \quad \pi_0 + \pi_1 = 1$$

From the first equation above, we get:

$$\pi_1 = \pi_0 \sqrt{x} + \frac{\pi_1}{2} \Rightarrow \pi_1 = 2\sqrt{x}\,\pi_0$$

and plugging this result in the second equation, we have:

$$\pi_0 (1 + 2\sqrt{x}) = 1 \Rightarrow \pi_0 = \frac{1}{1+2\sqrt{x}}$$

Also, $\pi_1 = \dfrac{2\sqrt{x}}{1+2\sqrt{x}}$

Finally, we know that every period spent at state 1 results in a profit of 0.75 units, while a period spent at state 0 incurs a cost of x. Hence, according to the theory presented in class, the ~~long~~-term expected profit rate that is mentioned in the problem can be expressed as

$$f(x) = 0.75\,\pi_1(x) - x\,\pi_2(x) = \frac{1.5\sqrt{x}}{1+2\sqrt{x}} - \frac{x}{1+2\sqrt{x}}$$

In order to find the extreme points of $f(x)$, we set

$$\frac{df(x)}{dx} = 0 \Rightarrow \frac{(0.75\,x^{-1/2} - 1)(1+2\sqrt{x}) - (1.5\sqrt{x} - x)\,x^{-1/2}}{(1+2\sqrt{x})^2} = 0 \Rightarrow$$

$$\Rightarrow \quad (0.75 - \sqrt{x})(1+2\sqrt{x}) - 1.5\sqrt{x} + x = 0 \Rightarrow$$

$$\Rightarrow \quad 0.75 - \sqrt{x} \cancel{+1.5\sqrt{x}} \cancel{-2x} \cancel{-1.5\sqrt{x}} \cancel{+x} = 0 \Rightarrow$$

$$\Rightarrow \quad x + \sqrt{x} - 0.75 = 0$$

set $\sqrt{x} \equiv y$. Then, we need to solve:

$$y^2 + y - 0.75 = 0 \xRightarrow[\text{since } y>0]{} y = \frac{-1 + \sqrt{1 + 4 \times 0.75}}{2} = \frac{-1+\sqrt{4}}{2} = \frac{2-1}{2} = 1/2$$

and $\sqrt{x} = 1/2 \Rightarrow x = 1/4$.

The corresponding profit rate is

$$f(1/4) = \frac{1.5(1/2)}{1+2(1/2)} - \frac{1/4}{1+2(1/2)} = \frac{0.75 - 0.25}{2} = \frac{0.5}{2} = \frac{1}{4}$$

Problem 5:

From the developments presented in class, we have:

$$a_1^{(t+1)} = a_n^{(t)}$$

$$a_2^{(t+1)} = \frac{V_1}{V_2} a_1^{(t)} + \left(1 - \frac{V_1}{V_2}\right) a_n^{(t)}$$

$$a_3^{(t+1)} = \frac{V_2}{V_3} a_2^{(t)} + \left(1 - \frac{V_2}{V_3}\right) a_n^{(t)}$$

$$\text{— — — — — — — —}$$

$$a_n^{(t+1)} = \frac{V_{n-1}}{V_n} a_{n-1}^{(t)} + \left(1 - \frac{V_{n-1}}{V_n}\right) a_n^{(t)}$$

From the above equations, we further obtain:

$$a_1^{(t+1)} = a_n^{(t)}$$

$$a_2^{(t+1)} = \frac{V_1}{V_2} a_n^{(t-1)} + \frac{V_2 - V_1}{V_2} a_n^{(t)}$$

$$a_3^{(t+1)} = \frac{V_2}{V_3}\left[\frac{V_1}{V_2} a_n^{(t-2)} + \left(1 - \frac{V_1}{V_2}\right) a_n^{(t-1)}\right] + \left(1 - \frac{V_2}{V_3}\right) a_n^{(t)}$$

$$= \frac{V_1}{V_3} a_n^{(t-2)} + \frac{V_2 - V_1}{V_3} a_n^{(t-1)} + \frac{V_3 - V_2}{V_3} a_n^{(t)}$$

More generally, for $i = 1, \dots, n$:

$$a_i^{(t+1)} = \frac{1}{V_i}\Big[V_1 a_n^{(t-i+1)} + (V_2 - V_1) a_n^{(t-i+2)} + (V_3 - V_2) a_n^{(t-i+3)} + \cdots$$
$$\cdots + (V_i - V_{i-1}) a_n^{(t)}\Big] \qquad (1)$$

In particular,

$$a_n^{(t+1)} = \frac{1}{V_n}\Big[V_1 a_n^{(t-n+1)} + (V_2 - V_1) a_n^{(t-n+2)} + \cdots + (V_n - V_{n-1}) a_n^{(t)}\Big] \quad (2)$$

If I can show that $\lim_{t\to\infty} a_n^{(t)} = a_n$, for some value $a_n \in \mathbb{R}$, then Eq. (1) implies that

$$\forall i, \quad \lim_{t\to\infty} a_i^{(t)} = \frac{1}{v_i}\big[\underbrace{v_1 + (v_2 - v_1) + (v_3 - v_2) + \ldots + (v_i - v_{i-1})}_{v_i}\big] a_n$$
$$= a_n$$

which is consistent with the results that were developed in class.

To show the convergence of $\{a_n^{(t)}, t = 1, 2 \ldots\}$, first we notice that from Eq. (2),

$a_n^{(t+1)}$ is ~~a~~ weighted average (or a convex combination) of the last n values of this sequence, i.e.,

$$a_n^{(t+1)} = \sum_{i=0}^{n-1} w_i\, a_n^{(t-i)}$$

where $\sum_{i=0}^{n-1} w_i = 1$, $w_i > 0\ \forall i$ (the exact values of w_i are defined by Eq. (2) but they are not necessary for showing the convergence of $\{a_n^{(t)}, t = 1, 2, \ldots\}$)

Set $\mathbf{v}_1^{(t)} = [a_n^{(t)}, a_n^{(t-1)}, \ldots, a_n^{(t-n+1)}]$

and also define $\text{range}(\mathbf{v}^{(t)}) = \max\{a_n^{(t)}, a_n^{(t-1)}, \ldots, a_n^{(t-n+1)}\} - \min\{a_n^{(t)}, a_n^{(t-1)}, \ldots, a_n^{(t-n+1)}\}$

To show the convergence of $\{a_n^{(t)}, t = 1, 2, \ldots\}$ as $t \to \infty$, it suffices to show that $\exists \gamma (0,1)$ such that $\text{range}(\mathbf{v}^{(t+n)}) \le \gamma \cdot \text{range}(\mathbf{v}^{(t)})$.

To establish this last result, consider for any $i = 0, 1, \ldots, n-1$, the distance:

$$\left| a_n^{(t+1)} - a_n^{(t-i)} \right| = \left| \sum_{j=0}^{n-1} w_j a_n^{(t-j)} - a_n^{(t-i)} \right| =$$

$$= \left| \sum_{j=0}^{n-1} w_j \left(a_n^{(t-j)} - a_n^{(t-i)} \right) \right| \leq \sum_{j=0}^{n-1} w_j \left| a_n^{(t-j)} - a_n^{(t-i)} \right|$$

$$= \sum_{\substack{j=0 \\ j\neq i}}^{n-1} w_j \left| a_n^{(t-j)} - a_n^{(t-i)} \right| \leq \left(\sum_{\substack{j=0 \\ j\neq i}}^{n-1} w_j \right) \max_{\substack{(i,j): \\ i<j}} \left| a_n^{(t-j)} - a_n^{(t-i)} \right|$$

$$= \left(\sum_{\substack{j=0 \\ j\neq i}}^{n-1} w_j \right) \text{range}\left(v^{(t)}\right) < \text{range}\left(v^{(t)}\right)$$

So, the replacement of $a_n^{(t-n+1)}$ in $v^{(t)}$ with $a_n^{(t+1)}$ in order to get $v^{(t+1)}$ will not increase the range of $v^{(t+1)}$ w.r.t. the range of $v^{(t)}$, and, in fact, it will decrease this range if $a_n^{(t-n+1)}$ was one of the components of $v^{(t)}$ that determined its range. Since every component of the current $v^{(t)}$ will be dropped from these vectors after n periods, we are guaranteed to have a decrease in the range of these vectors every n periods, as requested above.

Problem 4

(i) Since X has a finite state space and it is irreducible, it is also positive recurrent. And since it is also aperiodic, it is ergodic, i.e., it has a limiting distribution.

(ii)
$$q_a = \lim_{t\to\infty} P(Y_t = a) = \lim_{t\to\infty} P(X_t = 1 \vee X_t = 2) =$$
$$= \lim_{t\to\infty} P(X_t = 1) + \lim_{t\to\infty} P(X_t = 2) = \pi_1 + \pi_2$$
$$q_b = \lim_{t\to\infty} (Y_t = b) = \lim_{t\to\infty} P(X_t = 3) = \pi_3$$

(iii) Let the one-step transition probability matrix for the sought DTMC Z be denoted by
$$P_Z = \begin{bmatrix} q_{aa} & q_{ab} \\ q_{ba} & q_{bb} \end{bmatrix}$$

Since P_Z is a stochastic matrix, we need:
$$q_{aa}, q_{ab}, q_{ba}, q_{bb} \geq 0 \qquad (1)$$

and also
$$q_{aa} + q_{ab} = 1 \iff q_{aa} = 1 - q_{ab} \qquad (2)$$
$$q_{ba} + q_{bb} = 1 \iff q_{bb} = 1 - q_{ba} \qquad (3)$$

In addition, we must satisfy
$$(q_a \; q_b) = (q_a \; q_b) \begin{bmatrix} 1-q_{ab} & q_{ab} \\ q_{ba} & 1-q_{ba} \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} q_a(1-q_{ab}) + q_b q_{ba} = q_a \\ q_a q_{ab} + q_b(1-q_{ba}) = q_b \end{cases} \Rightarrow \begin{cases} -q_a q_{ab} + q_b q_{ba} = 0 \\ q_a q_{ab} - q_b q_{ba} = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \quad \frac{q_{ab}}{q_{ba}} = \frac{q_b}{q_a} \qquad (4)$$

From the above calculation, it follows that the sought matrix P_Z can be constructed as follows:

If $\frac{q_b}{q_a} \leq 1$ then pick some value $q_{ba} \in (0,1)$

Next, set $q_{ab} = \frac{q_b}{q_a} \cdot q_{ba}$. The selection of q_{ba} and the fact that $q_b / q_a \leq 1$ imply that $q_{ab} \in (0,1)$.

Finally, determine q_{aa} and q_{bb} from (2) and (3).

If $\frac{q_b}{q_a} > 1$, reverse the role of q_{ba} and q_{ab} in the above construction.

A particular set of values that will satisfy the above four conditions are

$$\begin{cases} q_{ab} = \dfrac{\pi_1}{\pi_1 + \pi_2} P_{13} + \dfrac{\pi_2}{\pi_1 + \pi_2} P_{23} \\[2ex] q_{ba} = P_{31} + P_{32} = \dfrac{\pi_3}{\pi_3} (P_{31} + P_{32}) \end{cases}$$

These are the probabilities that we shall observe the corresponding transitions in S_Y assuming that the DTMC X is initialized in S_X according to its limiting distribution π.

Then, we can easily check that this selection satisfies condition (0) and also (1) and (2) with $q_{aa} = \frac{\pi_1}{\pi_1 + \pi_2} P_{11}$ and $q_{bb} = P_{33}$.

On the other hand, Condition (4) can be checked as follows:

$$\frac{q_{ab}}{q_{ba}} = \frac{\pi_1 P_{13} + \pi_2 P_{23}}{(\pi_1 + \pi_2)(P_{31} + P_{32})} \overset{*}{=} \frac{\pi_3 (P_{31} + P_{32})}{(\pi_1 + \pi_2)(P_{31} + P_{32})} = \frac{\pi_3}{\pi_1 + \pi_2} = \frac{\varphi_b}{\varphi_a}$$

* This equality holds because at the equilibrium of the DTMC X, the total "inflow" to state 3 (i.e., $\pi_1 P_{13} + \pi_2 P_{23}$) is equal to the total "outflow" from this state (i.e., $\pi_3 (P_{31} + P_{32})$).

(iv) Since all these processes have a limiting distribution, the condition requested in this question is equivalent to the following one:

$$\pi_1 r_x(1) + \pi_2 r_x(2) + \pi_3 r_x(3) = \varphi_a r_y(a) + \varphi_b r_y(b) =$$

$$= (\pi_1 + \pi_2) r_y(a) + \pi_3 r_y(b) \Rightarrow$$

$$\Rightarrow r_y(b) = \frac{\pi_1}{\pi_3} r_x(1) + \frac{\pi_2}{\pi_3} r_x(2) + r_x(3) - \frac{\pi_1 + \pi_2}{\pi_3} r_y(a)$$

$$= \frac{1}{\pi_3}\left[\pi_1 r_x(1) + \pi_2 r_x(2) - (\pi_1 + \pi_2) r_y(a)\right] + r_x(3).$$

From the last equation, it should be clear that a pricing of r_y that will do the job, is

$$\begin{cases} r_y(a) = \dfrac{\pi_1}{\pi_1 + \pi_2} r_x(1) + \dfrac{\pi_2}{\pi_1 + \pi_2} r_x(2) \\[2ex] r_y(b) = r_x(3) \end{cases}$$

What is the intuitive interpretation of this result?