ISYE 7201: Production & Service Systems
Spring 2020
Instructor: Spyros Reveliotis
Final Exam (Take Home)
Release Date: April 20, 2020
Due Date: April 29, 2020

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

You can send me your responses as a pdf file attached to an email. This pdf file can be a scan of a hand-written document, but, please, write your answers very clearly and thoroughly. Also, report any external sources (other than your textbook) that you referred to while preparing the solutions.

Problem 1 (20 points): – An M/M/1 queue with reneging Consider an M/M/1 queue with arrival rate λ and processing rate μ, and the following modification in its operational dynamics: Each customer joining the station will *renege* (i.e., depart without service) with probability $ah + o(h)$ while waiting in the queue for a time interval of length h, independently of the other customers in the queue. But once a customer makes it to the server, she will receive full service and then depart from the system.

i. (5 pts) Show that the dynamics of this queueing station can be modeled as a continuous-time Markov chain (CTMC) with the state being the number of customers in the system. Provide a complete characterization of this CTMC.

ii. (5 pts) What is the stability condition for this CTMC?

iii. (5 pts) Consider the special case where $a = \mu$ (i.e., the customer reneging rate is equal to the server processing rate). Argue that in this case the CTMC will always be stable, and characterize the corresponding limiting distribution.

iv. (5 pts) For the special case defined in item (iii) above, what is the "steady-state" probability that a customer who gets into the station will renege?

Problem 2 (20 points): Consider a shop floor operating three identical machines. Each of these machines breaks down according to a Poisson law at an average rate of one every 10 hrs, and the failures are repaired one at a time by two maintenance technicians operating as two separate and identical servers, that serve the failing machines according to an exponential distribution with rate $\mu = 0.125\ hr^{-1}$ (think of this rate as one machine repair per one 8-hour shift).

i. (5 pts) Model the operating cycles of the three machines in the above facility by means of a two-station closed queueing network (CQN).

ii. (5 pts) Compute the "steady-state" probabilities

$$p_i \equiv P(\#\text{ of functional machines} = i),\ i = 0, 1, 2, 3$$

iii. (5 pts) What is the "mean time to repair (MTTR)" for any failing machine?

iv. (5 pts) Answer parts (i) – (iii) above assuming that the company acquires a fourth machine that is used, however, as a "spare"; i.e., when available, this machine will be used in the position of a failing machine, but, at any point in time, there will be no more than three machines operational in this shop-floor. For parts (ii) and (iii), you don't have to recompute everything; just describe a *plan of work* that will address these two questions in this new situation.

Problem 3 (20 points): – mean and variance of compound random variables Consider a random variable (r.v.) $Y = \sum_{i=1}^{N} X_i$, where $X_i, i = 1, 2, \ldots$, is a sequence of i.i.d. r.v.'s with mean $E[X]$ and variance $Var[X]$, and N is a discrete r.v. with mean $E[N]$ and variance $Var[N]$. Show that

- $E[Y] = E[N] \cdot E[X]$
- $Var[Y] = E[N] \cdot Var[X] + Var[N] \cdot E^2[X]$

Remark: To compute the above quantities, you can use the identity $E[Y] = E_N[E_X[Y|N]]$ and the identity $Var[Y] = E[Y^2] - E^2[Y]$. $E[Y^2]$ can be computed in a way similar to that suggested for the computation of $E[Y]$. The notation $E_Q[\cdot]$, $Q \in \{X, N\}$, implies that the expectation is taken with respect to the statistics of the corresponding r.v.

Problem 4 (20 points): – Bernoulli splitting of a renewal process Consider a renewal process generating a stream of events where the inter-event times $T_i,\ i = 1, 2, \ldots$, are i.i.d. r.v.'s with mean $E[T]$ and squared coefficient of variation $SCV[T]$. The events generated by this process are classified into K types with corresponding probabilities $p_k,\ k = 1, \ldots, K$, s.t. $\sum_{k=1}^{K} p_k = 1.0$. The generated events of type $k,\ k = 1, \ldots, K$, through this classification, define new renewal processes with inter-event times $T_i^{(k)},\ i = 1, 2, \ldots$. Sow that

- $E[T^{(k)}] = \frac{1}{p_k} E[T]$
- $Var[T^{(k)}] = \frac{1}{p_k} Var[T] + \frac{1-p_k}{p_k^2} E^2[T]$
- $SCV[T^{(k)}] = p_k \cdot SCV[T] + 1 - p_k$

Remark: For the first two parts of the above problem, use the results of Problem 3 in this exam. What is the distribution of the corresponding variable N in the considered context?

Problem 5 (20 points) – Approximate MVA Analysis of non-Markovian Open Queueing Networks through Kingman's approximation for the G/G/1 queue Consider a manufacturing cell with two single-server workstations, WS_1 and WS_2, where jobs arrive according to a Poisson process with rate $r_a = 10\ hr^{-1}$. The processing of these jobs starts at workstation WS_1 where they execute a first processing stage, and subsequently they move to workstation WS_2 for the execution of a second processing stage. But the execution of this second processing stage by a part at workstation WS_2 is successful only with probability 0.8. Successfully processed parts leave the cell, and this completes the processing of the corresponding job. In the opposite case, the current part is scrapped, and a new part is initiated at workstation WS_1, to replace the original one.

For this cell, please, do the following:

i. (5 pts) Represent the workflow taking place in this cell as a queueing network, and compute the total *part* arrival rate $\lambda_i,\ i = 1, 2$, for each workstation.

ii. (5 pts) Determine the mean processing time for the server of each workstation so that each server has a utilization level of 90%.

iii. (5 pts) What is the departure rate of the *completed jobs* from this cell under the mean processing times that you computed in part (ii) above?

iv. (5 pts) How many parts must be processed, on average, in order to get a job completed?

v. (5 pts) Assuming that the server mean processing time at each workstation is that computed in part (ii) above, and the coefficient of variation (CV) of the processing times at each of the two workstations is 0.5, compute the average number of parts that are in each of the two workstations.

Remark: Notice that this problem has considerable similarity with the example on the MVA of open QNs that was presented in our lectures, the primary difference being that the dynamics that concern the processing that takes place at each station are not Markovian any more. In this case, in order to answer questions like those posed in part (v) of this problem, we can use the results of the MVA for the G/G/1 queue that is based on Kingman's approximation. In order to pursue such an analysis, notice that part (v) itself specifies the mean processing time and the CV for these processing times, for each workstation. Also, you will have the total part arrival rates, for each workstation, through part (i) of the problem. But for the application of Kingman's approximation to each of these stations, you will also need an estimate of the SCV of the part inter-arrival times for each workstation. Try to obtain these estimates using the results of Problem 4 in this exam, and the following additional result that provides an approximate estimate of the SCV, c_a^2, of the inter-arrival times for an arrival stream that merges the arrivals from two independent sub-streams with arrival rates $\lambda_i,\ i = 1, 2$, and SCVs for their inter-arrival times equal to $c_{a_i}^2$; we have:

$$c_a^2 \approx \frac{\lambda_1}{\lambda_1 + \lambda_2} c_{a_1}^2 + \frac{\lambda_2}{\lambda_1 + \lambda_2} c_{a_2}^2 \tag{1}$$

Using the aforementioned results, try to set up and solve a linear system of equations, in a spirit similar to that of the traffic equations, that will give you the required estimates.

Also notice that the result of Eq. 1 subsumes the case where the two merged arrival streams are Poisson. In this case, $c_{a_i} = 1.0$ for both streams, and therefore, according to this equation, $c_a = 1.0$, as well. This is in agreement with the previously established result that the counting process that results from the merging of two independent Poisson processes is also Poisson. Finally, the result of Eq. 1 generalizes to the case of merging more than two independent arrival streams in the natural manner. You can also see Section 5.3 in the textbook on Advanced Manufacturing Systems Modeling and Analysis by Curry and Feldman for more discussion on this issue.

Problem 1

(i) Working in a way similar to that used for the modeling of the various Markovian queueing stations discussed in the lectures by a CTMC, we can see that we can model the considered queueing station by a CTMC with the following STD:

(ii) The above CTMC is a birth-death process with arrival rate λ and state-dependent proc. rates; in particular, $\mu(n) = \mu + (n-1)a$ for $n = 1, 2, \ldots$

Then, the stability condition is

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{M(n)} < \infty \quad \text{where } M(n) = \begin{cases} \mu(1)\mu(2)\ldots\mu(n), & \text{for } n \geq 1 \\ 1, & \text{for } n = 0. \end{cases}$$

For the $\mu(n)$ function that was defined above, we have:

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{M(n)} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{\mu(\mu+a)\cdots(\mu+(n-1)a)} =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(\lambda/a)^n}{\frac{\mu}{a}\left(\frac{\mu}{a}+1\right)\cdots\left(\frac{\mu}{a}+n-1\right)} = 1 + \frac{\lambda}{\mu} \sum_{n=1}^{\infty} \frac{(\lambda/a)^{n-1}}{\left(\frac{\mu}{a}+1\right)\cdots\left(\frac{\mu}{a}+n-1\right)}$$

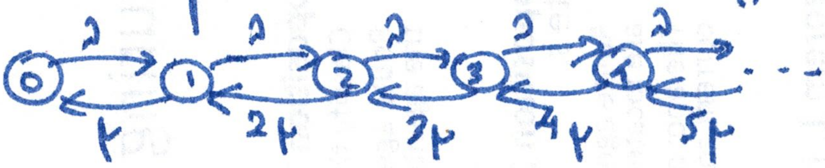
$$= 1 + \frac{\lambda}{\mu} \sum_{n=0}^{\infty} \frac{(\lambda/a)^n}{\left(\frac{\mu}{a}+1\right)\cdots\left(\frac{\mu}{a}+n\right)} < 1 + \frac{\lambda}{\mu} \sum_{n=0}^{\infty} \frac{(\lambda/a)^n}{n!} =$$

$$= 1 + \frac{\lambda}{\mu} e^{\lambda/a} < \infty$$

Hence, this station will always be stable.

An intuitive interpretation of this result is that the queue tends to explode, the rate with which customers will renege also increases, keeping the queue stable eventually.

(iii) In the special case where $a = \mu$, we get the CTMC:



This CTMC is identical to the CTMC of an $M/M/\infty$ queue with arrival rate λ and server proc. rate μ.

We know that this last queue is always stable (which is consistent with our earlier finding for the new queue) and has a Poisson limiting distribution with parameter $\rho = \lambda/\mu$.

(iv) The "steady-state" throughput for the considered queue is $TH = (1 - p_0)\mu$, where from part (iii), $p_0 = e^{-\rho}$;

hence, $TH = (1 - e^{-\rho})\mu$.

Then, Prob (a joining customer completes service) =

$$= \frac{TH}{\lambda} = (1 - e^{-\rho})\frac{\mu}{\lambda} = \frac{1 - e^{-\rho}}{\rho}$$

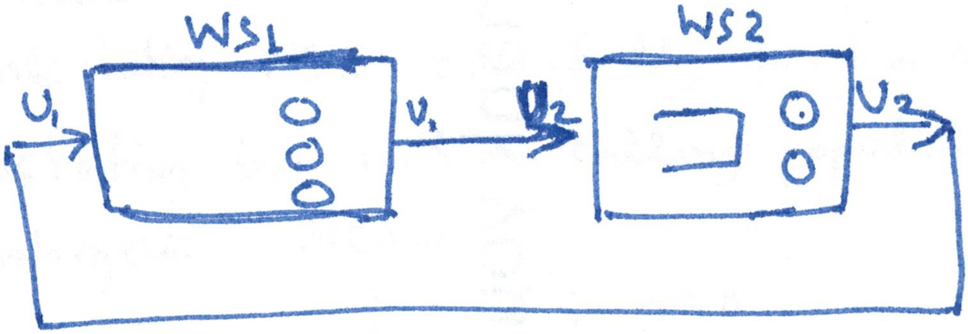
and

Prob (a joining customer reneges) =

$$= 1 - \frac{1 - e^{-\rho}}{\rho} = \frac{\rho - 1 + e^{-\rho}}{\rho}.$$

Problem 2

(i) We can model the operating cycles of the three machines by a two-station CQN where the first station, WS1, models the operation of these machines during their "uptime", and the second station, WS2, models the "downtime" phase of these machines. Schematically, this network will look as follows:



It is obvious from the cyclical structure of the above network that the relative arrival rates at the two stations, v_1 and v_2, must satisfy $v_1 = v_2$ (of course, you can get this result from the corresponding traffic equations). In the following we shall take $v_1 = v_2 = 1$

To model the experience of the three machines in their uptime and downtime phases, it suffices to define accordingly the process-rate functions $\mu_i(n)$, $i = 1, 2$; $n = 0, 1, 2, 3$, for the two workstations.

More specifically:

A) For workstation WS_1 we will have

$$\mu_1(n) = \begin{cases} 0; & n=0 \\ 0.1\ hr^{-1}; & n=1 \\ 0.2\ hr^{-1}; & n=2 \\ 0.3\ hr^{-1}; & n=3 \end{cases}$$

The above definition of $\mu_1(n)$ reflects the fact that each operational machine fails independently from the other ones and the time to its failure is exp. distributed with rate $0.1\ hr^{-1}$.

B) Workstation WS2 essentially function as an M/M/2 workstation but with a calling population of no more than 3 customers. Hence

$$\mu_2(n) = \begin{cases} 0; & n=0 \\ 1/8 = 0.125\ hr^{-1}; & n=1 \\ 2/8 = 0.25\ hr^{-1}; & n=2 \\ 2/8 = 0.25\ hr^{-1}; & n=3 \end{cases}$$

(ii) – (iii) To answer questions (ii) and (iii) we have to execute the recursive algorithm for the MVA of CQNs that we presented in the lectures.

In the execution of this algorithm, we shall take $\underline{V_1 = V_2 = 1}$.

Also, the notation used in the following is that used in the corresponding lecture notes.

The "boundary" condition for the recursion that is necessary for our needs, is:

$$P[X_i = \emptyset;\ \emptyset] = 1 \quad ; \quad i = 1, 2.$$

Then, for $\underline{N=1}$:

$$W_i(1) = \sum_{n=1}^{1} \frac{n}{\mu_i(n)} P[X_i = n-1; \emptyset] = \frac{1}{\mu_i(1)} P[X_i = \emptyset; \emptyset]$$

$$\Rightarrow \begin{cases} W_1(1) = \frac{1}{0.1} \cdot 1 = 10 \text{ hrs} \\ W_2(1) = \frac{1}{1/8} \cdot 1 = 8 \text{ hrs} \end{cases}$$

$$TH(1) = \frac{1}{\sum_{i=1}^{2} U_i W_i(1)} = \frac{1}{W_1(1) + W_2(1)} = \frac{1}{10+8} = \frac{1}{18} = 0.0\bar{5} \text{ hr}^{-1}$$

$$\begin{cases} P[X_i = n; 1] = \frac{U_i\, TH(1)}{\mu_i(n)} P[X_i = n-1; \emptyset] & i = 1,2;\ n = 1 \\ P[X_i = \emptyset; 1] = 1 - \sum_{n=1}^{1} P[X_i = n; 1] \end{cases}$$

Hence,

$$\begin{cases} P[X_1 = 1; 1] = \frac{1}{18} \cdot \frac{1}{0.1} \cdot 1 = 0.\bar{5} \simeq 0.56 \\ P[X_1 = \emptyset; 1] = 1 - 0.56 \simeq 0.44 \end{cases}$$

$$\begin{cases} P[X_2 = 1; 1] = \frac{1}{18} \cdot 8 \cdot 1 = \frac{8}{18} \simeq 0.44 \\ P[X_2 = 0; 1] = 1 - 0.44 = 0.56 \end{cases}$$

Notice that, as expected, these numbers add up to 1.0

for $N=2$:

$$W_i(2) = \sum_{n=1}^{2} \frac{n}{\mu_i(n)} P[X_i = n-1; 1] =$$

$$= \frac{1}{\mu_i(1)} P[X_i = 0; 1] + \frac{2}{\mu_i(2)} P[X_i = 1; 1] \Rightarrow$$

$$\Rightarrow \begin{cases} W_1(2) = \frac{1}{0.1} \cdot 0.44 + \frac{2}{0.2} 0.56 = 10 \text{ hr} \\ W_2(2) = 1/(1/8) \cdot 0.56 + 2/(2/8) \cdot 0.44 = 8 \text{ hr} \end{cases}$$

$$TH(2) = \frac{2}{\sum_{i=1}^{2} v_i W_i(2)} = \frac{2}{W_1(2) + W_2(2)} = \frac{2}{10+8} = \frac{1}{9} = 0.\bar{1} \text{ hr}^{-1}$$

$$\begin{cases} P[X_i = n; 2] = \frac{v_i TH(2)}{\mu_i(n)} P[X_i = n-1; 1], \; i=1,2; \; n=1,2 \\ P[X_i = 0; 2] = 1 - \sum_{n=1}^{2} P[X_i = n; 2] \end{cases}$$

Hence,

$$\begin{cases} P[X_1 = 1; 2] = \frac{1}{9} \cdot \frac{1}{0.1} \cdot 0.44 \simeq 0.49 \\ P[X_1 = 2; 2] = \frac{1}{9} \cdot \frac{1}{0.2} \cdot 0.56 \simeq 0.31 \\ P[X_1 = 0; 2] = 1 - 0.49 - 0.31 = 0.2 \end{cases}$$

$$\begin{cases} P[X_2 = 1; 2] = \frac{1}{9} \cdot 8 \cdot 0.56 \simeq 0.5 \\ P[X_2 = 2; 2] = \frac{1}{9} \cdot \frac{8}{2} \cdot 0.44 \simeq 0.2 \\ P[X_2 = 0; 2] = 1 - 0.5 - 0.2 = 0.3 \end{cases}$$

Finally, for N=3

$$W_i(3) = \sum_{n=1}^{3} \frac{n}{\mu_i(n)} P[X_i = n-1; 2] =$$

$$= \frac{1}{\mu_i(1)} P[X_i = \emptyset; 2] + \frac{2}{\mu_i(2)} P[X_i = 1; 2] + \frac{3}{\mu_i(3)} P[X_i = 2; 2]$$

$$\Rightarrow \begin{cases} W_1(3) = \frac{1}{0.1} 0.2 + \frac{2}{0.2} 0.49 + \frac{3}{0.3} 0.31 = 10 \text{ hrs} \\ W_2(3) = \frac{1}{1/8} \cdot 0.3 + \frac{2}{2/8} \cdot 0.5 + \frac{3}{2/8} \cdot 0.2 = 8{,}8 \text{ hrs} \end{cases}$$

(iii)

$$TH(3) = \frac{3}{\sum_{i=1}^{2} v_i W_i(3)} = \frac{3}{W_1(3) + W_2(3)} = \frac{3}{10 + 8.8} = 0{,}16 \text{ hr}^{-1}$$

finally, we can get the distribution that is requested in part (ii):

$$\begin{cases} P[X_1 = n; 3] = \frac{v_1 TH(3)}{\mu_1(n)} P[X_1 = n-1; 2], \ n = 1,2,3 \\ P[X_1 = 0; 3] = 1 - \sum_{n=1}^{3} P[X_1 = n; 3] \end{cases}$$

$$\Rightarrow \begin{cases} P[X_1 = 1; 3] = 0.16 \cdot \frac{1}{0.1} \cdot 0.2 = 0.32 \\ P[X_1 = 2; 3] = 0.16 \cdot \frac{1}{0.2} \cdot 0.49 = 0{,}392 \\ P[X_1 = 3; 3] = 0.16 \cdot \frac{1}{0.3} \cdot 0.31 \simeq 0.165 \\ P[X_1 = 0; 3] \ 1 - 0.32 - 0.392 - 0.165 = 0.123 \end{cases}$$

(iv) The introduction of the fourth machine in this shop-floor, to function as a "spare" machine, will not change the basic topology of the modeling CAN that was defined in part (i). However, now $N = 4$, and the proc. rate functions for the two workstations WS_1 and WS_2 must be revised as follows:

$$\mu_1(n) = \begin{cases} 0 \; ; & n = 0 \\ 0.1 \text{ hr}^{-1} \; ; & n = 1 \\ 0.2 \text{ hr}^{-1} \; ; & n = 2 \\ 0.3 \text{ hr}^{-1} \; ; & n \in \{3, 4\} \end{cases}$$

$$\mu_2(n) = \begin{cases} 0 \; ; & n = 0 \\ 1/8 \; ; & n = 1 \\ 2/8 \; ; & n \in \{2, 3, 4\} \end{cases}$$

Then, the questions in parts (ii) and (iii) of the problem can be answered for this new situation using the revised values for the problem parameters.

Problem 3

We have:

$$E[Y] = E_N\left[E_X[Y \mid N]\right] = E_N\left[E_X\left[\sum_{i=1}^{N} X_i \mid N\right]\right] =$$

$$= E_N\left[N \cdot E[X]\right] = E[N]\, E[X]$$

Also,

$$E[Y^2] = E_N\left[E_X[Y^2 \mid N]\right] = E_N\left[E_X\left[\left(\sum_{i=1}^{N} X_i\right)^2 \mid N\right]\right] =$$

$$= E_N\left[E_X\left(\sum_{i=1}^{N} X_i^2 + \sum_{i=1}^{N}\sum_{j \neq i}^{N} X_i X_j \mid N\right)\right] =$$

$$= E_N\left[N \cdot E[X^2] + N(N-1)\, E^2[X]\right] =$$

$$= E[N] \cdot E[X^2] + E[N^2]\, E^2[X] - E[N]\, E^2[X]$$

Furthermore,

$$Var[Y] = E[Y^2] - E^2[Y] =$$

$$= E[N]\, E[X^2] + E[N^2]\, E^2[X] - E[N]\, E^2[X]$$

$$- E^2[N]\, E^2[X] =$$

$$= E[N]\left(E[X^2] - E^2[X]\right) +$$

$$+ E^2[X]\left(E[N^2] - E^2[N]\right) =$$

$$= E[N] \cdot Var[X] + E^2[X] \cdot Var[N]$$

Problem 4.

Let r.v $T^{(k)}$ = the time between two consecutive events of type k.

Then,
$$T^{(k)} = \sum_{i=1}^{N} T_i$$

where r.v. N is the number of events from the original renewal process that ~~ta~~ take place after the occurrence of the first event of type k, until we get the second event from this type. Also the r.v.'s T_i, $i=1,\dots,N$, denote the corresponding inter-event times in the original renewal process.

Then, $T^{(k)}$ has the structure of the compound r.v.'s defined in Problem 3. Applying the results of that problem we have:

- $E[T^{(k)}] = E[N] \cdot E[T]$
- $Var[T^{(k)}] = E[N] \cdot Var(T) + Var(N) \, E^2[T]$

In the considered setting, each of the i events for $i \in \{1,\dots,N\}$ can be perceived as a Bernoulli trial with success probability p_k. Then, r.v. N counts the number of trials till the first success, and therefore, it follows a geometric distribution with parameter p_k.

Hence, $E[N] = 1/p_k$ and $Var[N] = \frac{1-p_k}{p_k^2}$

Plugging these values of $E[N]$ & $Var[N]$ to the above expressions of $E[T^{(k)}]$ and $Var[T^{(k)}]$, we get:

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$$- E[T^{(k)}] = \frac{1}{p_k} E[T]$$

$$- Var[T^{(k)}] = \frac{1}{p_k} Var[T] + \frac{1-p_k}{p_k^2} E^2[T]$$

Finally,

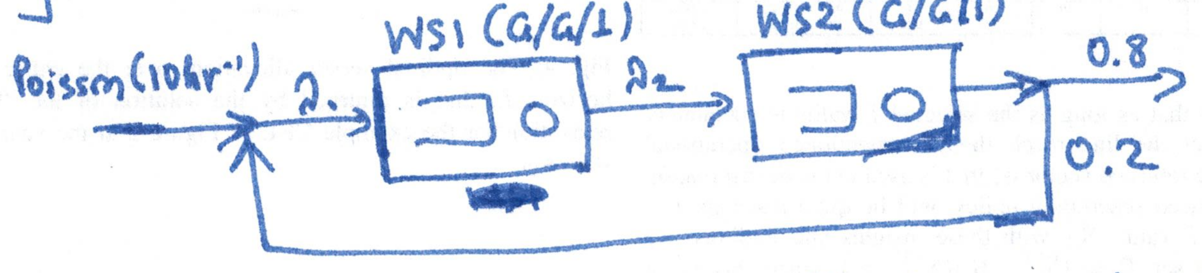
$$SCV[T^{(k)}] = \frac{Var[T^{(k)}]}{E^2[T^{(k)}]} =$$

$$= \frac{\frac{1}{p_k} Var[T] + \frac{1-p_k}{p_k^2} E^2[T]}{\frac{1}{p_k^2} E^2[T]} =$$

$$= p_k \frac{Var[T]}{E^2[T]} + 1 - p_k = p_k \cdot SCV[T] + 1 - p_k$$

Problem 5

(i) Since a scrapped part initiates a new part at WS_1 that is necessary in order to fill the corresponding order, we can model the operation of this cell through the following open network:



The above network is not a Jackson network since the distributions that characterize the processing times at the servers of the two workstations are not necessarily exponential (see part (v) below). But we can still write down the traffic equations that must be observed by this network when operated in stable mode, as in the case of Jackson networks; in particular, letting λ_i, $i=1,2$ denote the total arrival rates for the two workstations, we will have:

$$\begin{cases} \lambda_1 = 10 + 0.2\lambda_2 \\ \lambda_2 = \lambda_1 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = 12.5 \text{ hr}^{-1}.$$

(ii) Letting t_i, $i=1,2$ denote the mean processing time at WS_i we must have:

$$\lambda_i \cdot t_i = 0.9 \Rightarrow t_i = 0.9/\lambda_i = 0.9/12.5 = 0.072 \text{ hrs}$$

(iii) Since all jobs are completed, even if some initial attempts for each of these jobs are scrapped, and the system is stable, it must be that the departure rate of these jobs is equal to the corresponding arrival rate of 10 jobs/hr.

An alternative argument to obtain this result is that good parts leave the cell with rate of

$$0.8 \times \lambda_2 = 0.8 \times 12.5 = 10 \text{ hr}^{-1}.$$

But good parts correspond to completed jobs.

(iv) Each initiated part at station WS_1 is a Bernoulli trial to fill the corresponding job, and the success probability of these trials is 0.8. Hence, the expected number of trials till success is equal to

$$\frac{1}{0.8} = 1.25.$$

Another way to get this result is by computing the ratio of $\lambda_1/10 = \frac{12.5}{10} = 1.25$, since the increased arrival rate of parts at station WS_1 w.r.t. the external arrival rate of jobs is due exactly to the fact that each of these jobs might need the initiation of more than one parts until it is successfully completed.

(v) We can get an estimate of the average number of parts at each of the two workstations, by using the formulae for the approximate MVA of the $G/G/1$ queue that were discussed in the last lecture. According to these formulae we have:

$$L_i = E[\#\text{ of parts at } WS_i] = \lambda_i W_{q_i} + u_i \quad (1)$$

where u_i = server utilization at station WS_i and

$W_{q_i} = E[$ waiting time in the buffer of station $WS_i]$

The values of u_i and λ_i are already known, and from Kingman's approximation we have:

$$W_{q_i} \simeq \frac{c_{a_i}^2 + c_{p_i}^2}{2} \frac{u_i}{1-u_i} \cdot t_i \quad (2)$$

where - $c_{a_i}^2 \equiv$ the SCV of the total arrival stream at station WS_i and

- $c_{p_i}^2 \equiv$ the SCV of the processing times at station WS_i

The c_{p_i} values are given by the problem but the required values of $c_{a_i}^2$ must be computed. For this, we shall work as suggested in the remark that follows Problem 5.

Let us also denote the CV of the inter-departure times from station WS_1 by C_{d_1}, and the CV for the inter-event times for the sequence of the part transfers from station WS_2 back to station WS_1, by C_{21}. Then, we have the following equations:

$$C_{a_1}^2 = \frac{10}{\lambda_1} \cdot 1 + \frac{0.2 \cdot \lambda_2}{\lambda_1} \cdot C_{12}^2 =$$

$$= \frac{10}{12.5} + 0.2\, C_{12}^2 = 0.8 + 0.2\, C_{12}^2 \quad (3)$$

The first term in the rhs of the above equation recognizes that the inter-event times for the Poisson arrival process are exponentially distributed, and therefore, the corresponding SCV is equal to 1.0.

$$C_{12}^2 = 0.2 \cdot C_{d_2}^2 + 1 - 0.2 = 0.2\, C_{d_2}^2 + 0.8 \quad (4)$$

$$C_{d_2}^2 = U_2^2 \cdot C_{p_2}^2 + (1 - U_2^2)\, C_{a_2}^2 =$$

$$= 0.9^2 \cdot 0.5^2 + (1 - 0.9^2) \cdot C_{a_2}^2$$

$$= 0.2025 + 0.19\, C_{a_2}^2 \quad (5)$$

$$C_{a_2}^2 = C_{d_1}^2 = U_1^2 \cdot C_{p_1}^2 + (1 - U_1^2) \cdot C_{a_1}^2 =$$

$$= 0.9^2 \cdot 0.5^2 + (1 - 0.9^2)\, C_{a_1}^2 =$$

$$= 0.2025 + 0.19\, C_{a_1}^2 \quad (6)$$

Equations (3) - (6) constitute a linear system of four equations in the four unknowns $C_{a_1}^2$, C_{12}^2, $C_{d_2}^2$ and $C_{a_2}^2$. Solving it, we get:

$$C_{a_1}^2 \simeq 0.971 \quad \text{and} \quad C_{a_2}^2 \simeq 0.387$$

Then, from (2) we have:

$$W_{q_1} \simeq \frac{0.971 + 0.5^2}{2} \cdot \frac{0.9}{1-0.9} \cdot 0.072 \simeq 0.3956 \text{ hrs}$$

$$W_{q_2} \simeq \frac{0.387 + 0.5^2}{2} \cdot \frac{0.9}{1-0.9} \cdot 0.072 \simeq 0.2064 \text{ hrs}$$

and

$$L_1 = 12.5 \times 0.3956 + 0.9 = 5.845$$

$$L_2 = 12.5 \times 0.2064 + 0.9 = 3.48$$

It is interesting to notice that even though these two stations have the same first and second moments for their proc. times and the same ~~total~~ arrival rates λ_i (and therefore, the same server utilizations u_i), the congestion levels experienced at each of these two stations are different because of the difference of the variability that is experienced in the arrival processes at each of the two stations.

At station WS_1, the dominant element that defines the variability in the arrival process experienced by this station is the Poisson process of the external arrivals, which has considerable variability. On the other hand, the arrival process at WS_2 is defined by the departure process of WS_1 and the variability of this process is defined primarily by the variability of the proc. times at station WS_1, since the utilization of the

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station server is quite high (0.9). But the CV for the proc. time distribution for this station is only 0.5, considerably lower than 1.0.