STOCHASTIC PROCESSES

Second Edition

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Taking expectations gives

$$E[A(t)] = a_0 + E\left[\int_0^t X(s)\,ds\right]$$
$$= a_0 + \int_0^t E[X(s)]\,ds \quad \text{since } X(s) \geq 0$$
$$= a_0 + \int_0^t e^{\lambda s}\,ds$$
$$= a_0 + \frac{e^{\lambda t} - 1}{\lambda}.$$

The following example provides another illustration of a pure birth process.

Example 5.3(c) ***A Simple Epidemic Model.*** Consider a population of m individuals that at time 0 consists of one "infected" and $m - 1$ "susceptibles." Once infected an individual remains in that state forever and we suppose that in any time interval h any given infected person will cause, with probability $\alpha h + o(h)$, any given susceptible to become infected. If we let $X(t)$ denote the number of infected individuals in the population at time t, the $\{X(t), t \geq 0\}$ is a pure birth process with

$$\lambda_n = \begin{cases} (m-n)n\alpha & n = 1, \ldots, m-1 \\ 0 & \text{otherwise.} \end{cases}$$

The above follows since when there are n infected individuals, then each of the $m - n$ susceptibles will become infected at rate $n\alpha$.

If we let T denote the time until the total population is infected, then T can be represented as

$$T = \sum_{i=1}^{m-1} T_i,$$

where T_i is the time to go from i infectives to $i + 1$ infectives. As the T_i are independent exponential random variables with respective rates $\lambda_i = (m - i)i\alpha$, $i = 1, \ldots, m - 1$, we see that

$$E[T] = \frac{1}{\alpha}\sum_{i=1}^{m-1} \frac{1}{i(m-i)}$$

and

$$\text{Var}(T) = \frac{1}{\alpha^2}\sum_{i=1}^{m-1}\left(\frac{1}{i(m-i)}\right)^2.$$

For reasonably sized populations $E[T]$ can be approximated as follows:

$$E[T] = \frac{1}{m\alpha}\sum_{i=1}^{m-1}\left(\frac{1}{m-i} + \frac{1}{i}\right)$$
$$\approx \frac{1}{m\alpha}\int_1^{m-1}\left(\frac{1}{m-t} + \frac{1}{t}\right)dt = \frac{2\log(m-1)}{m\alpha}.$$

5.4 The Kolmogorov Differential Equations

Recall that

$$P_{ij}(t) = P\{X(t+s) = j \mid X(s) = i\}$$

represents the probability that a process presently in state i will be in state j a time t later.

By exploiting the Markovian property, we will derive two sets of differential equations for $P_{ij}(t)$, which may sometimes be explicitly solved. However, before doing so we need the following lemmas.

Lemma 5.4.1

(i) $\lim_{t\to 0} \frac{1 - P_{ii}(t)}{t} = v_i.$

(ii) $\lim_{t\to 0} \frac{P_{ij}(t)}{t} = q_{ij}, \quad i \neq j.$

Lemma 5.4.2

For all s, t,

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s).$$

Lemma 5.4.1 follows from the fact (which must be proven) that the probability of two or more transitions in time t is $o(t)$; whereas Lemma 5.4.2, which is the continuous-time version of the Chapman-Kolmogorov equations of discrete-time Markov chains, follows directly from the Markovian property. The details of the proof are left as exercises.

From Lemma 5.4.2 we obtain

$$P_{ij}(t+h) = \sum_k P_{ik}(h)P_{kj}(t),$$

or, equivalently,

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k\neq i} P_{ik}(h)P_{kj}(t) - [1 - P_{ii}(h)]P_{ij}(t).$$

Dividing by h and then taking the limit as $h \to 0$ yields, upon application of Lemma 5.4.1,

(5.4.1) $$\lim_{h\to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lim_{h\to 0} \sum_{k\neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) - v_i P_{ij}(t).$$

Assuming that we can interchange the limit and summation on the right-hand side of (5.4.1), we thus obtain, again using Lemma 5.4.1, the following.

THEOREM 5.4.3 (Kolmogorov's Backward Equations).

For all i, j, and $t \geq 0$,

$$P'_{ij}(t) = \sum_{k\neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

Proof To complete the proof we must justify the interchange of limit and summation on the right-hand side of (5.4.1). Now, for any fixed N,

$$\liminf_{h\to 0} \sum_{k\neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) \geq \liminf_{h\to 0} \sum_{\substack{k\neq i\\ k<N}} \frac{P_{ik}(h)}{h} P_{kj}(t)$$
$$= \sum_{\substack{k\neq i\\ k<N}} q_{ik} P_{kj}(t).$$

Since the above holds for all N we see that

(5.4.2) $$\liminf_{h\to 0} \sum_{k\neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) \geq \sum_{k\neq i} q_{ik} P_{kj}(t).$$

To reverse the inequality note that for $N > i$, since $P_{kj}(t) \leq 1$,

$$\limsup_{h\to 0} \sum_{k\neq i} \frac{P_{ik}(h)}{h} P_{kj}(t)$$
$$\leq \limsup_{h\to 0} \left[\sum_{\substack{k\neq i\\ k<N}} \frac{P_{ik}(h)}{h} P_{kj}(t) + \sum_{k\geq N} \frac{P_{ik}(h)}{h} \right]$$
$$= \limsup_{h\to 0} \left[\sum_{\substack{k\neq i\\ k<N}} \frac{P_{ik}(h)}{h} P_{kj}(t) + \frac{1 - P_{ii}(h)}{h} - \sum_{\substack{k\neq i\\ k<N}} \frac{P_{ik}(h)}{h} \right]$$
$$= \sum_{\substack{k\neq i\\ k<N}} q_{ik} P_{kj}(t) + v_i - \sum_{\substack{k\neq i\\ k<N}} q_{ik},$$

where the last equality follows from Lemma 5.4.1. As the above inequality is true for all $N > i$, we obtain upon letting $N \to \infty$ and using the fact $\sum_{k\neq i} q_{ik} = v_i$,

$$\limsup_{h\to 0} \sum_{k\neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) \leq \sum_{k\neq i} q_{ik} P_{kj}(t).$$

The above combined with (5.4.2) shows that

$$\lim_{h\to 0} \sum_{k\neq i} \frac{P_{ik}(h)}{h} P_{kj}(t) = \sum_{k\neq i} q_{ik} P_{kj}(t),$$

which completes the proof of Theorem 5.4.3.

The set of differential equations for $P_{ij}(t)$ given in Theorem 5.4.3 are known as the Kolmogorov *backward equations*. They are called the backward equations because in computing the probability distribution of the state at time $t + h$ we conditioned on the state (all the way) back at time h. That is, we started our calculation with

$$P_{ij}(t+h) = \sum_k P\{X(t+h) = j \mid X(0) = i, X(h) = k\} P\{X(h) = k \mid X(0) = i\}$$
$$= \sum_k P_{kj}(t) P_{ik}(h).$$

We may derive another set of equations, known as the Kolmogorov's *forward equations*, by now conditioning on the state at time t. This yields

$$P_{ij}(t+h) = \sum_k P_{ik}(t) P_{kj}(h)$$

or

$$P_{ij}(t+h) - P_{ij}(t) = \sum_k P_{ik}(t)P_{kj}(h) - P_{ij}(t)$$
$$= \sum_{k \neq j} P_{ik}(t)P_{kj}(h) - [1 - P_{jj}(h)]P_{ij}(t).$$

Therefore,

$$\lim_{h\to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lim_{h \to 0} \left\{ \sum_{k\neq j} P_{ik}(t) \frac{P_{kj}(h)}{h} - \frac{1 - P_{jj}(h)}{h} P_{ij}(t) \right\}.$$

Assuming that we can interchange limit with summation, we obtain by Lemma 5.4.1 that

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Unfortunately, we cannot always justify the interchange of limit and summation, and thus the above is not always valid. However, they do hold in most models—including all birth and death processes and all finite-state models. We thus have

THEOREM 5.4.4 (Kolmogorov's Forward Equations).

Under suitable regularity conditions,

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Example 5.4(a) ***The Two-State Chain.*** Consider a two-state continuous-time Markov chain that spends an exponential time with rate λ in state 0 before going to state 1, where it spends an exponential time with rate μ before returning to state 0. The forward equations yield

$$P'_{00}(t) = \mu P_{01}(t) - \lambda P_{00}(t)$$
$$= -(\lambda + \mu)P_{00}(t) + \mu,$$

where the last equation follows from $P_{01}(t) = 1 - P_{00}(t)$. Hence,

$$e^{(\lambda+\mu)t}[P'_{00}(t) + (\lambda+\mu)P_{00}(t)] = \mu e^{(\lambda+\mu)t}$$

or

$$\frac{d}{dt}[e^{(\lambda+\mu)t}P_{00}(t)] = \mu e^{(\lambda+\mu)t}.$$

Thus,

$$e^{(\lambda+\mu)t}P_{00}(t) = \frac{\mu}{\lambda+\mu} e^{(\lambda+\mu)t} + c.$$

Since $P_{00}(0) = 1$, we see that $c = \lambda/(\lambda + \mu)$, and thus

$$P_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}.$$

Similarly (or by symmetry),

$$P_{11}(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}.$$

Example 5.4(b) The Kolmogorov forward equations for the birth and death process are

$$P'_{i0}(t) = \mu_1 P_{i1}(t) - \lambda_0 P_{i0}(t),$$
$$P'_{ij}(t) = \lambda_{j-1}P_{i,j-1}(t) + \mu_{j+1}P_{i,j+1}(t) - (\lambda_j + \mu_j)P_{ij}(t), \qquad j \neq 0.$$

Example 5.4(c) For a pure birth process, the forward equations reduce to

$$P'_{ii}(t) = -\lambda_i P_{ii}(t),$$
$$P'_{ij}(t) = \lambda_{j-1}P_{i,j-1}(t) - \lambda_j P_{ij}(t), \qquad j > i. \tag{5.4.3}$$

Integrating the top equation of (5.4.3) and then using $P_{ii}(0) = 1$ yields

$$P_{ii}(t) = e^{-\lambda_i t}.$$

The above, of course, is true as $P_{ii}(t)$ is the probability that the time until a transition from state i is greater than t. The other