ISYE 7201: Production & Service Systems
Spring 2019
Instructor: Spyros Reveliotis
2nd Midterm Exam (Take Home)
Release Date: March 25, 2019
Due Date: April 1, 2019

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

You can send me your responses as a pdf file attached to an email. This pdf file can be a scan of a hand-written document, but, please, write your answers very clearly and thoroughly. Also, report any external sources (other than your textbook) that your referred to while preparing the solutions.

Finally, Homework 2 and 3 posted at the course website, as well as some of the past midterms also posted at that website, can be provide useful complementary material (and experience) to the in-class lectures while working on the exam problems.

Problem 1 (20 points): If $X_i, i = 1, 2, 3$, are independent exponential random variables with rates $\lambda_i, i = 1, 2, 3$, compute

i. $P[X_1 < X_2 < X_3]$;

ii. $P[X_1 < X_2 \mid \max\{X_1, X_2, X_3\} = X_3]$;

iii. $E[\max X_i \mid X_1 < X_2 < X_3]$;

iv. $E[\max X_i]$.

Problem 2 (20 points): Consider a game that is played as follows: Events occur according to a Poisson process with rate λ. Each time an event occurs, we must decide whether to stop or not, with our objective being to stop at the last event that will occur before some specified time τ. This time τ is defined with respect to the initiation of the current round of the game (i.e., at the initiation of the current round, time is reset to zero), and it also holds that $\tau > 1/\lambda$. If no event occurs during the interval $(0, \tau]$ or if we stop prematurely (i.e., at an event that is not the last event in the interval $(0, \tau]$, then, we loose; otherwise, we win.

We want to play a strategy where we shall stop at the first event that will occur after some fixed time $s \in [0, \tau]$. You task is to choose the value of s in a way that maximizes the probability of winning, and also to compute the corresponding probability.

Problem 3 (20 points): Potential customers arrive at a full-service, one-pump gas station at a Poisson rate of 20 cars per hour. However, customers will only enter the station for gas if there are no more than two cars already in it (including the car that gets serviced). Also, suppose that the time required to service a car is exponentially distributed with a mean of five minutes. Determine the following:

i. What fraction of the attendant's time will be spent servicing cars?

ii. What fraction of potential customers are lost?

Problem 4 (20 points): Consider a manufacturing workstation that operates as follows: Parts to be processed are drawn from a local 'raw-materials' inventory according to a non-idling policy, and each part is processed as follows:

1. The part first goes through a processing stage with a processing time that follows a normal distribution with a mean of 10 minutes and a st. deviation of 3 minutes.
2. After the completion of the first processing stage, the part is inspected with the corresponding inspection time being normally distributed with mean 4 minutes and st. deviation 2 minutes. Furthermore, it has been observed that 70% of the parts pass successfully this inspection step, another 5% has to be scrapped, and the remaining parts must be reworked through an additional processing stage.
3. The processing times of the rework stage are uniformly distributed from 5 to 15 minutes (the rework process is stopped if it exceeds this time), and the success rate of this rework step is 80%. A part that fails this additional processing stage is scrapped.

Model the operation of the considered workstation as a semi-Markov process, and use this representation to determine the effective throughput of this workstation.

Problem 5 (20 points): Consider a state i of a CTMC $\{X(t),\ t \geq 0\}$, where the corresponding service time T_i is determined by the "exponential race" between two independent events e_k and e_l with corresponding instantaneous rates λ_k and λ_l. Furthermore, let k and l also denote the states of $\{X(t)\}$ that result from the execution of the corresponding events at state i. Prove or disprove the following:

$$P(T_i \leq t \mid i \rightarrow k) \ = \ P(T_i \leq t)$$

In the above expression, '$i \rightarrow k$' denotes the event that the process transitioned from state i to state k.

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Solutions to Midterm II

Problem 1:

(i) $P[X_1 < X_2 < X_3] = P[X_1 = \min\{X_1, X_2, X_3\}] \cdot P[X_2 < X_3 \mid X_1 = \min\{X_1, X_2, X_3\}]$

$$= \frac{\lambda}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3}$$

Alternatively, we have:

$P[X_1 < X_2 < X_3] = P[X_1 < \min\{X_2, X_3\} \wedge \min\{X_2 X_3\} = X_2] =$

$= P[X_1 < \min\{X_2, X_3\}] \cdot P[X_2 = \min\{X_2, X_3\}] =$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3}$$

(ii) $P[X_1 < X_2 \mid \max\{X_1, X_2, X_3\} = X_3] =$

$$= \frac{P[X_1 < X_2 \wedge \max\{X_1, X_2, X_3\} = X_3]}{P[\max\{X_1, X_2, X_3\} = X_3]} =$$

$$= \frac{P[X_1 < X_2 < X_3]}{P[X_1 < X_2 < X_3] + P[X_2 < X_1 < X_3]} =$$

$$= \frac{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3}}{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3} + \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_1}{\lambda_1+\lambda_3}} = \frac{1}{1 + \frac{\lambda_2+\lambda_3}{\lambda_1+\lambda_3}} =$$

$$= \frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3}$$

(iii) $E[\max X_i \mid X_1 < X_2 < X_3] =$

$$= E[\min\{X_1, X_2, X_3\}] + E[\min\{X_2, X_3\}] + E[X_3] =$$

$$= \frac{1}{\lambda_1+\lambda_2+\lambda_3} + \frac{1}{\lambda_2+\lambda_3} + \frac{1}{\lambda_3}$$

(iv) $E[\max_i] = \sum_{(p,q,r)} E[\max X_i \mid X_p < X_q < X_r] \cdot P[X_p < X_q < X_r]$

where the triplets (p,q,r) are all the possible (6) permutation of $(1,2,3)$.

Each term in the above sum can be computed from the result of parts (i) and (iii).

Problem 2:

A schematic representation of the situation that is described in this problem is as follows:

We shall win at any given round of the game only if there is one event occurring in the interval (s, τ). From the theory of the Poisson process, the corresponding probability $P(\text{win})$ is:

$$P(\text{win}) = e^{-\lambda(\tau-s)}\lambda(\tau-s).$$

To select s so that this probability is maximized, we set:

$$\frac{dP(\text{win})}{ds} = 0 \Rightarrow e^{-\lambda(\tau-s)}\lambda^2(\tau-s) + e^{-\lambda(\tau-s)}(-\lambda) = 0 \Rightarrow$$

$$\Rightarrow e^{-\lambda(\tau-s)}\lambda[\lambda(\tau-s)-1] = 0 \Rightarrow \tau - s = 1/\lambda \Rightarrow$$

$$\Rightarrow s = \tau - 1/\lambda.$$

Since it is assumed that $\tau > 1/\lambda$, the above value for s is a valid solution, i.e., $s \in (0, \tau)$.

To verify that the computed extreme point is indeed a maximum, we consider:

$$\frac{d^2P(\text{win})}{ds^2} = e^{-\lambda(\tau-s)}\lambda^2[\lambda(\tau-s)-1] - e^{-\lambda(\tau-s)}\lambda^3.$$

For $s = \tau - 1/\lambda$:

$$\left.\frac{d^2 P(\text{win})}{ds^2}\right|_{s=\tau-1/\lambda} = -\lambda^3 e^{-1} < 0$$

and therefore, the considered point is indeed a maximum.

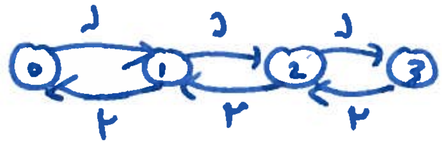
Finally, the maximal probability of winning that is obtained by the above value for s, is:

$$P^*(\text{win}) = e^{-1} \simeq 0.37$$

Problem 3:

This is essentially an $M/M/1/3$ queueing station with $\lambda = 20\ hr^{-1}$, $\mu = 12\ hr^{-1}$, and therefore $\rho = \lambda/\mu = \frac{5}{3} \simeq 1.67$.

The CTMC that models the dynamics of this queue is:



and it is known to have an equilibrium distribution $\pi_i, i = 0,1,2,3$.

Also, we have:

$$\forall i, \quad \pi_i = \pi_0 \rho^i \quad \text{and} \quad \sum_{i=0}^{3} \pi_i = 1 \Rightarrow \pi_0 (1 + \rho + \rho^2 + \rho^3) = 1 \Rightarrow$$

$$\Rightarrow \pi_0 = \frac{1}{1 + 1.67 + 1.67^2 + 1.67^3} \simeq 0.1$$

Thus,

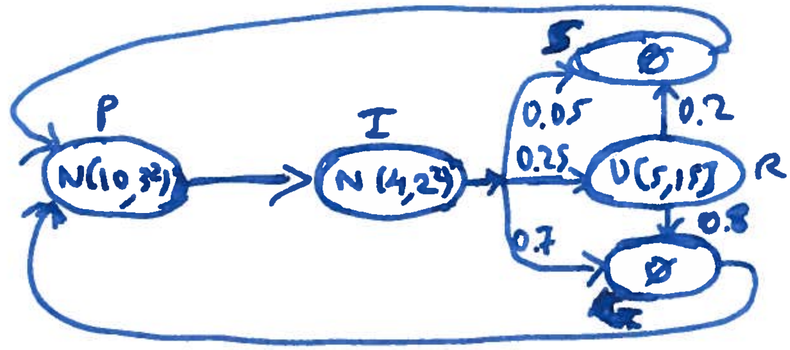
(i) $u = 1 - \pi_0 = 1 - 0.1 = 0.9$

(ii) $\pi_3 = \pi_0 \rho^3 = 0.1 \times 1.67^3 = 0.466$

and this number is the requested fraction thanks to PASTA.

Problem 4:

We can model the processing of a single part in this workstation through the following semi-Markov process:



The states of this process are defined as follows:

P: first processing stage

I: Inspection after first proc. stage

R: Rework

G: Output of a good part

S: " " " scrapped "

The information written in each node is the distribution of the corresponding sojourn times. For nodes G and S these sojourn times are deterministically equal to ∅, since the applying non-idling policy implies that another part will be immediately started, drawing the necessary input from the "raw materials" inventory. This fact is further reflected through the transitions from these two states back to state P.

Based on the above representation of the station operation, the average rate with which parts are loaded into this workstation is

$$R_L = \frac{1}{\text{mean recurrence time of state } P} \quad (1)$$

But since not all started parts make it to a successful output unit, the effective throughput of the station is

$$TH = R_L \cdot \text{Prob (successful processing)}. \quad (2)$$

In this case, the necessary quantities for the above computation can be easily obtained through inspection of the defined process. Hence, we have:

Mean recurrence time of state $P = 10 + 4 + 0.25 \times 10 = 16.5$ min.

Prob (successful processing) $= 0.7 + 0.25 \times 0.8 = 0.9$

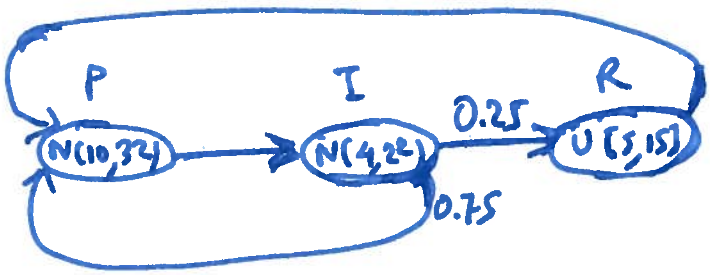
So,

$$R_L = \frac{1}{16.5 \text{ min}} \simeq 0.06 \text{ min}^{-1} \simeq 3.63 \text{ hr}^{-1}$$

and

$$TH = \frac{1}{16.5 \text{min}} \times 0.9 = 0.054 \text{ min}^{-1} \simeq 3.27 \text{ hr}^{-1}$$

Next, we also provide an alternative computation of the mean recurrence time of state P that relates more directly to the corresponding theory of the semi-Markov processes that was presented in class. In order to expedite this computation, we also use the following simplified model for the underlying semi-Markov process:



Let $\pi = (\pi_P, \pi_I, \pi_R)$ denote the limiting distribution for the embedded DTMC of the above process, and r.v. T_{PP} denote the recurrence time of state P in this (continuous-time) process. Also, let τ_x, $x = P, I, R$, denote the mean sojourn times for the different states. In class we showed that:

$$\forall x, \quad \pi_x E[T_{xx}] = \sum_x \pi_x \tau_x$$

Therefore: $$E[T_{PP}] = \tau_P + \frac{\pi_I}{\pi_P}\tau_I + \frac{\pi_R}{\pi_P}\tau_R =$$

$$= 10 + \frac{\pi_I}{\pi_P} 4 + \frac{\pi_R}{\pi_P} 10 \qquad (3)$$

Next, we compute the ratios $\frac{\pi_I}{\pi_P}$ and $\frac{\pi_R}{\pi_P}$ that appear in the above expression.

The one-step transition prob. matrix of the embedded DTMC of the simplified semi-Markov process is:

$$\hat{P} = \begin{array}{c} \\ P \\ I \\ R \end{array} \begin{array}{c} \begin{array}{ccc} P & I & R \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0.75 & 0 & 0.25 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Hence, we have:

$$[\pi_P \ \pi_I \ \pi_R] = [\pi_P \ \pi_I \ \pi_R] \begin{bmatrix} 0 & 1 & 0 \\ 0.75 & 0 & 0.25 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} \pi_P = 0.75\,\pi_I + \pi_R \\ \pi_I = \pi_P \end{cases} \Rightarrow \begin{cases} \pi_I/\pi_P = 1 \\ \pi_R/\pi_P = 1 - 0.75\,\frac{\pi_I}{\pi_P} = 0.25 \end{cases} \quad (4)$$

Plugging the above values in (3), we get:

$$E[T_{PP}] = 10 + 4 + 0.25 \times 10 = 16.5 \text{ min}.$$

We can also use the simplified semi-Markov process to organize an alternative computation of the effective throughput as follows:

Let p_x, $x = P, I, R$, denote the limiting distribution for the semi-Markov process. Then, reasoning as in the corresponding example presented in class, we can write:

$$TH = p_I \cdot \frac{1}{\tau_I} \cdot 0.7 + p_R \cdot \frac{1}{\tau_R} \cdot 0.8 \qquad (5)$$

Also, we have: $\quad \forall x, \quad p_x = \dfrac{\pi_x \tau_x}{\sum_y \pi_y \tau_y} \qquad (6)$

From (5) and (6):

$$TH = \frac{\pi_I \cdot 0.7 + \pi_R \cdot 0.8}{\pi_P \tau_P + \pi_I \tau_I + \pi_R \tau_R} = \frac{(\pi_I/\pi_P) \cdot 0.7 + (\pi_R/\pi_P) \cdot 0.8}{\tau_P + (\pi_I/\pi_P)\tau_I + (\pi_R/\pi_P)\tau_R} =$$

$$\overset{(3),(4)}{=} \frac{0.7 + 0.25 \times 0.8}{E[T_{PP}]}$$

which is exactly the formula that we have derived in the previous computation of TH.

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(iii) The processing times of the rework stage are uniformly distributed from 5 to 15 minutes (the rework process is stopped if it exceeds this time), and the success rate of this rework step is 80%. A part that fails this additional processing stage is scrapped.

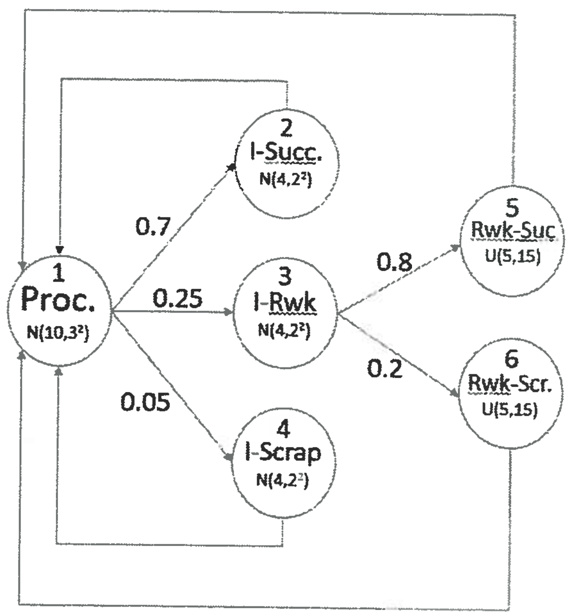
Model the operation of the considered workstation as a semi-Markov process, and use this representation to determine the effective throughput of this workstation.

Solution:

A graphic of the considered workstation modeled as a semi-Markov process and it's corresponding embedded DTMC transition probability matrix are below.

An alternative solution for Problem 4 from one of your fellow students.

This solution is closer to the spirit of the model for the corresponding problem that I presented in class.



State	τ_i
1	10
2	4
3	4
4	4
5	10
6	10

$$\hat{\mathbb{P}} = \begin{bmatrix} 0 & 0.7 & 0.25 & 0.05 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving for π:

$$\pi_2 = 0.7\pi_1$$
$$\pi_3 = 0.25\pi_1$$
$$\pi_4 = 0.05\pi_1$$
$$\pi_5 = 0.8\pi_3$$
$$\pi_6 = 0.2\pi_3$$
$$1 = \pi_1 + 0.7\pi_1 + 0.25\pi_1 + 0.05\pi_1 + 0.8(0.25\pi_1) + 0.2(0.25\pi_1)$$
$$\Longrightarrow \pi^T = \left[\frac{40}{90}, \frac{28}{90}, \frac{10}{90}, \frac{2}{90}, \frac{8}{90}, \frac{2}{90}\right]$$

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To find the effective throughput of the workstation, both states 2 and 5 need to be analyzed, as shown below.

$$
\begin{aligned}
p_i &= \frac{\pi_i\tau_i}{\sum_{k=1}^{6}\pi_k\tau_k}\\
p_2 &= \frac{\pi_2\tau_2}{\sum_{k=1}^{6}\pi_k\tau_k}\\
&= \frac{(\frac{28}{90})4}{(\frac{40}{90})10+(\frac{28}{90})4+(\frac{10}{90})4+(\frac{2}{90})4+(\frac{8}{90})10+(\frac{2}{90})10}\\
&= \frac{28}{165}\\
p_5 &= \frac{\pi_5\tau_5}{\sum_{k=1}^{6}\pi_k\tau_k}\\
&= \frac{(\frac{8}{90})10}{(\frac{40}{90})10+(\frac{28}{90})4+(\frac{10}{90})4+(\frac{2}{90})4+(\frac{8}{90})10+(\frac{2}{90})10}\\
&= \frac{4}{33}\\
\text{TP}_2 &= p_2(\frac{1}{\tau_2}) * 60 \text{ minutes}\\
&= 60\left(\frac{28}{165}\right)\left(\frac{1}{4}\right)\\
&= \frac{84}{33}\\
&\approx 2.54/\text{hour}\\
\text{TP}_5 &= p_5(\frac{1}{\tau_5}) * 60 \text{ minutes}\\
&= 60\left(\frac{4}{33}\right)\left(\frac{1}{10}\right)\\
&= \frac{24}{33}\\
&\approx 0.73/\text{hour}\\
\text{TP}_{\text{eff}} &\approx 3.3/\text{hour}
\end{aligned}
$$

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5. Consider a state i of a CTMC $X(t), t \geq 0$, where the corresponding service time T_i is determined by the "exponential race" between two independent events e_k and e_l with corresponding instantaneous rates λ_k and λ_l. Furthermore, let k and l also denote the states of $X(t)$ that result from the execution of the corresponding events at state i. Prove or disprove the following:

$$P(T_i \leq t | i \to k) = P(T_i \leq t)$$

Solution:

$$
\begin{aligned}
P(T_i \leq t|i \to k) &= 1 - P(T_i > t|i \to k)\\
&= 1 - P(T_i > t|T_{e_k} < T_{e_l})\\
&\text{where } T_{e_k} < T_{e_l} \text{ because } e_k \text{ happened before } e_l\\
&= 1 - \frac{P(T_i > t, T_{e_k} < T_{e_l})}{P(T_{e_k} < T_{e_l})}\\
&= 1 - \frac{P(t < T_{e_k} < T_{e_l})}{P(T_{e_k} < T_{e_l})}\\
&= 1 - \frac{\left(\frac{\lambda_k}{\lambda_k+\lambda_l}\right)e^{-(\lambda_k+\lambda_l)t}}{\left(\frac{\lambda_k}{\lambda_k+\lambda_l}\right)}\\
&= 1 - e^{-(\lambda_k+\lambda_l)t}\\
&= P(T_i \leq t)
\end{aligned}
$$

Problem 5:

The structure that is described in this problem can be depicted as follows:

$$
\begin{tikzcd}
 & k \\
i \arrow[ur, "\lambda_k"] \arrow[dr, "\lambda_e"'] & \\
 & e
\end{tikzcd}
$$

Then, we have:

$$P(T_i \leq t \mid i \to k) = \frac{P(T_i \leq t \wedge i \to k)}{P(i \to k)} = \frac{P(\min\{T_{ek}, T_{ee}\} \leq t \wedge T_{ek} < T_{e}}{P(T_{ek} < T_{ee})} \quad ①$$

where the r.v.'s T_{ek} and T_{ee} denote the times till the occurrence of the corresponding events. The probabilities in the right-hand-side of the above equation can be computed as follows:

$$P(T_{ek} < T_{ee}) = \frac{\lambda_k}{\lambda_k + \lambda_e} \quad ②$$

$$P(\min\{T_{ek}, T_{ee}\} \leq t \wedge T_{ek} < T_{ee}) = P(T_{ek} \leq t \wedge T_{ee} > T_{ek}) =$$

$$= \int_0^t e^{-\lambda_e x} \lambda_k e^{-\lambda_k x} \, dx = \frac{\lambda_k}{\lambda_k + \lambda_e} [1 - e^{-(\lambda_k + \lambda_e)t}] \quad ③$$

From ①, ② and ③ we have:

$$P[T_i \leq t \mid i \to k] = 1 - e^{-(\lambda_k + \lambda_e)t} = P[T_i \leq t]$$