ISYE 7201: Production & Service Systems
Spring 2019
Instructor: Spyros Reveliotis
1st Midterm Exam (Take Home)
Release Date: January 30, 2019
Due Date: February 6, 2019

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

You can send me your responses as a pdf file attached to an email. This pdf file can be a scan of a hand-written document, but, please, write your answers very clearly and thoroughly. Also, report any external sources (other than your textbook) that your referred to while preparing the solutions.

Finally, Homework 1 posted at the course website, as well as some of the past midterms also posted at that website, can be provide useful complementary material (and experience) to the in-class lectures while working on the exam problems.

Problem 1 (30 points): Let $\{X_n\}$ and $\{Y_n\}$ be two independent Markov chains, each with the same discrete state space $S = \{0, 1, 2\}$ and with the same probability matrix

$$\begin{pmatrix} .3 & .3 & .4 \\ .2 & .7 & .1 \\ .2 & .3 & .5 \end{pmatrix}$$

Define the process $\{Z_n\} = \{(X_n, Y_n)\}$ with state space $S \times S$.

i. **(10 pts)** Argue formally that the process $\{Z_n\}$ is a Markov chain, and define the corresponding one-step transition probability matrix.

ii. **(20 pts)** Also, suppose that process $\{Z_n\}$ is initialized at state $(0, 1)$ and compute the expected time until the process finds itself at a state (X_n, Y_n) with $X_n = Y_n$.

Problem 2 (20 points): Consider a Markov chain on states $\{0, 1, 2\}$ with one-step transition probability matrix

$$\begin{pmatrix} .3 & .3 & .4 \\ .2 & .7 & .1 \\ .2 & .3 & .5 \end{pmatrix}$$

Compute the probability $P[X_{12} = 2, X_{16} = 2, X_{20} = 2 | X_0 = 2]$. What is the natural interpretation of this quantity?

Problem 3 (20 points): Consider an irreducible Markov chain with state space $S = \{1, \ldots, m\}$ and one-step transition probability matrix P, that is also *doubly stochastic*; i.e.,

$$\sum_{i \in S} p_{ij} = 1, \ \forall j \in S$$

.

i. **(10 pts)** Show that the (row) vector $\pi = (1/m, \ldots, 1/m)$ is a stationary distribution for this chain; i.e.,

$$\pi = \pi \cdot P \ ; \ \pi \cdot \mathbf{1} = 1$$

where $\mathbf{1}$ denotes the m-dimensional (column) vector with all of its components equal to 1.

ii. **(10 pts)** Furthermore, show that if the considered Markov chain is also aperiodic, then, for any vector $\mathbf{v} \in R^m$,

$$\lim_{n \to \infty} P^n \cdot \mathbf{v} \;=\; \mu \mathbf{1}$$

where

$$\mu = \frac{1}{m} \sum_{i=1}^{m} \mathbf{v}[i]$$

Can you think of any practical applications of this last result?

Problem 4 (30 points): Read the paper

- J. J. Bartholdi and D. D. Eisenstein, *"A production line that balances itself"*, OR, vol. 44, no. 1, 1996

and provide an account of (i) the main results of this paper, and (ii) the way that the Perron-Frobenius theorem and its adaptation to stochastic matrices that were discussed in class, facilitate the development of those results. In fact, for the purposes of this exam, you can focus the more technical part of the requested discussion on the key result of Theorem 3 in the aforementioned paper and the simplified model that underlies this result. But, please, try to be as thorough and lucid as possible in your discussion of these developments.

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SPRING 2019
SOLUTIONS TO MIDTERM I

Problem 1:

(i) We have:

$$\text{Prob}\{(X_{k+1}, Y_{k+1}) = (i_{k+1}, j_{k+1}) \mid (X_0,Y_0)=(i_0,j_0), \ldots, (X_k,Y_k)=(i_k,j_k)\}$$

$$= \text{Prob}\{X_{k+1} = i_{k+1} \mid X_0 = i_0, \ldots, X_k = i_k\} \cdot \text{Prob}\{Y_{k+1} = j_{k+1} \mid Y_0 = j_0, \ldots, Y_k = j_k\}$$

$$= \text{Prob}\{X_{k+1} = i_{k+1} \mid X_k = i_k\} \cdot \text{Prob}\{Y_{k+1} = j_{k+1} \mid Y_k = j_k\} =$$

$$= \text{Prob}\{(X_{k+1}, Y_{k+1}) = (i_{k+1}, j_{k+1}) \mid (X_k, Y_k) = (i_k, j_k)\}$$

The first and third equalities in the above derivation result from the independence of $\{X_n\}$ and $\{Y_n\}$, and the second one from their Markovian nature.

The derivation itself shows that the process $\{(X_n, Y_n)\}$ is Markov.

Also, we can see that

$$P_{(ij)(kl)} \equiv \text{Prob}\{(X_1,Y_1)=(k,l) \mid (X_0,Y_0)=(i,j)\} =$$

$$= \text{Prob}\{X_1 = k \mid X_0 = i\} \cdot \text{Prob}\{Y_1 = l \mid Y_0 = j\} =$$

$$= P_{ik} \cdot P_{jl}$$

Finally using the last result of the previous page, the matrix P that collects the one-step transition probabilities for chain $\{Z_n\}$ can be obtained as follows:

$$
P = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (0,2) \\ (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} (0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) & (2,0) & (2,1) & (2,2) \end{array} \\
\begin{bmatrix}
0.09 & 0.09 & 0.12 & 0.09 & 0.09 & 0.12 & 0.12 & 0.12 & 0.16 \\
0.06 & 0.21 & 0.03 & 0.06 & 0.21 & 0.03 & 0.08 & 0.28 & 0.04 \\
0.06 & 0.09 & 0.15 & 0.06 & 0.09 & 0.15 & 0.08 & 0.12 & 0.20 \\
0.06 & 0.06 & 0.08 & 0.21 & 0.21 & 0.28 & 0.03 & 0.03 & 0.04 \\
0.04 & 0.14 & 0.02 & 0.14 & 0.49 & 0.07 & 0.02 & 0.07 & 0.01 \\
0.04 & 0.06 & 0.10 & 0.14 & 0.21 & 0.35 & 0.02 & 0.03 & 0.05 \\
0.06 & 0.06 & 0.08 & 0.09 & 0.09 & 0.12 & 0.15 & 0.15 & 0.20 \\
0.04 & 0.14 & 0.02 & 0.06 & 0.21 & 0.03 & 0.10 & 0.35 & 0.05 \\
0.04 & 0.06 & 0.10 & 0.06 & 0.09 & 0.15 & 0.10 & 0.15 & 0.25
\end{bmatrix}
\end{array}
$$

It is also interesting to observe the "symmetries" that are present in the above matrix and are due to the facts that (i) the two composed processes $\{X_n\}$ and $\{Y_n\}$ are independent and (ii) evolve on the same state space with the same one-step transition prob. matrix.

Hence, due to these symmetries we have:

$$P_{(i,j)(k,\ell)} = P_{(j,i)(\ell,k)}$$

Is there any additional such "symmetry" present in the above matrix?

(ii) We can compute this expected time by "collapsing" the three target states (0,0), (1,1), (2,2) into a single absorbing state to be denoted by '$*$'. The one-step transition prob. matrix for the resulting process can be written as follows:

$$
\begin{array}{c}
 \\ (0,1) \\ (0,2) \\ (1,0) \\ (1,2) \\ (2,0) \\ (2,1) \\ *
\end{array}
\begin{array}{c}
\begin{array}{cccccc|c} (0,1) & (0,2) & (1,0) & (1,2) & (2,0) & (2,1) & * \end{array} \\
\left[\begin{array}{cccccc|c}
0.21 & 0.03 & 0.06 & 0.03 & 0.08 & 0.28 & 0.31 \\
0.09 & 0.15 & 0.06 & 0.15 & 0.08 & 0.12 & 0.35 \\
0.06 & 0.08 & 0.21 & 0.28 & 0.03 & 0.03 & 0.31 \\
0.06 & 0.10 & 0.14 & 0.35 & 0.02 & 0.03 & 0.30 \\
0.06 & 0.08 & 0.09 & 0.12 & 0.15 & 0.15 & 0.35 \\
0.14 & 0.02 & 0.06 & 0.03 & 0.01 & 0.35 & 0.30 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

Then, as discussed in class, letting P_T denote the upper-left (6×6)-dim submatrix of the above matrix, the expected absorption times, f_{ij}, from each state (i,j) with $i \neq j$, to state $*$ can be computed by the formula:

$$\underline{f} = (I - P_T)^{-1}\, \underline{1}$$

In the above formula:

* $\underline{f}$ is the vector that collects the quantities $f(i,j)^{i \neq j}$, in the order that they appear in matrix P_T.
* I is the (6×6)-dim identity matrix.
* $\underline{1}$ is the 6-dim vector with all its components equal to 1.

Using MATLAB to perform the above computation, we obtain

$$f = [3.2065,\ 3.0592,\ 3.2065,\ 3.2393,\ 3.0592,\ 3.2393]^T$$

and the particular "absorption" time that is requested by the problem, is $f(0,1) = f[1] = 3.2065$.

<u>Remark:</u> Notice that the aforementioned "symmetries" appear also in the content of vector f.

<u>Problem 2:</u>

We have:

$$P[X_{12}=2, X_{16}=2, X_{20}=2 \mid X_0=2] =$$

$$= P[X_{20}=2 \mid X_0=2, X_{12}=2, X_{16}=2] \cdot P[X_{12}=2, X_{16}=2 \mid X_0=2] =$$

$$= P[X_{20}=2 \mid X_{16}=2] \cdot P[X_{16}=2 \mid X_0=2, X_{12}=2] \cdot P[X_{12}=2 \mid X_0=2] =$$

$$= P[X_4=2 \mid X_0=2] \cdot P[X_{16}=2 \mid X_{12}=2] \cdot P[X_{12}=2 \mid X_0=2] =$$

$$= (P[X_4=2 \mid X_0=2])^2 \cdot P[X_{12}=2 \mid X_0=2]$$

Let P denote the provided one-step trans. prob. matrix.

Then
$$P^4 = \begin{bmatrix} 0.2223 & 0.4872 & 0.2905 \\ 0.2222 & 0.5128 & 0.2650 \\ 0.2222 & 0.4872 & 0.2906 \end{bmatrix}$$

and $P[X_4=2 \mid X_0=2] = (P^4)[3,3] = 0.2906$.

Also,

$$P^{12} = (P^4)^3 = \begin{bmatrix} 0.222 & 0.5 & 0.2778 \\ 0.222 & 0.5 & 0.2778 \\ 0.222 & 0.5 & 0.2778 \end{bmatrix}$$

and $P[X_{12}=2 \mid X_0=2] = (P^{12})[3,3] = 0.2778.$

Finally,

$$P[X_{12}=2, X_{16}=2, X_{20}=2 \mid X_0=2] = (0.2906)^2 \cdot 0.2778 =$$
$$= 0.02346.$$

This is the probability that the considered MC will start at state 2, and at periods 12, 16 and 20 will be again at the same state.

Remark: It is also interesting to notice that by period 12, the matrix P^{12} has already reached the limit $\lim_{k\to\infty} P^k$. In fact, even the matrix P^4 is not too far away from this limit.

Problem 3:

(i) Obviously $\pi \cdot 1 = \sum_{i=1}^{m} \frac{1}{m} \cdot 1 = \frac{m}{m} = 1$.

Also, let $P[\cdot, j]$, $j = 1, \dots, m$, denote the j-th column of the one-step trans. prob. matrix of the considered MC. Then, $\forall j = 1, \dots, m$:

$$\pi \cdot P[\cdot, j] = [1/m, 1/m, \dots, 1/m] \begin{bmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{mj} \end{bmatrix} =$$

$$= \frac{1}{m} \sum_{i=1}^{m} p_{ij} = \frac{1}{m} \cdot 1 = \frac{1}{m}$$

The next-to-last equality results from the fact that matrix P is doubly stochastic. Also, the above computation establishes that $\pi = \pi \cdot P$, and therefore π is a stationary distribution.

(ii) If the considered chain is irreducible and aperiodic, then it has a unique stationary distribution that is also a limiting distribution for this chain. In particular, for $n \to \infty$, P^n converges to a matrix P^∞ where the rows of this matrix are equal to the unique stationary distribution π. From part (i), we know that $\pi = [1/m, 1/m, \dots, 1/m]$. The above remarks further imply that, for any vector $v \in R^m$:

$$\lim_{n \to \infty} P^n \cdot v = P^\infty \cdot v = \begin{bmatrix} 1/m & 1/m & \dots & 1/m \\ 1/m & 1/m & \dots & 1/m \\ \vdots & \vdots & & \vdots \\ 1/m & 1/m & \dots & 1/m \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{m}\sum_{i=1}^{m} v_i \\ \frac{1}{m}\sum_{i=1}^{m} v_i \\ \cdots \\ \frac{1}{m}\sum_{i=1}^{m} v_i \end{bmatrix} = \left(\frac{1}{m}\sum_{i=1}^{m} v_i\right) \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix} = \mu \cdot \underline{1}$$

Remark: The above result has been proposed as an effective mechanism for computing, in a distributed manner, the average of a set of values $\{v_i, i=1, \dots, m\}$ that have been obtained independently by a set of "agents" a_i, $i=1,\dots,m$.

In the proposed mechanism, each agent a_i "possesses" the parameters of the i-th row of some primitive doubly stochastic matrix P, and it can communicate directly only with agents a_j for which the corresponding entry p_{ji} in matrix P is non-zero (the communication is unidirectional; i.e., agent a_i can send information to agent a_j, but a_j cannot communicate with agent a_i — on the other hand, this last communication is possible if p_{ij} is also non-zero).

The overall computation evolves through a number of "rounds", where at each round:

(i) each agent a_i communicates its current value v_i to its neighboring agents a_j (i.e, those with $p_{ji} \neq 0$), and subsequently

(ii) each agent a_i updates its current value v_i through the following computation: $v_i := p_{ii} v_i + \sum_{j:\, p_{ij} \neq 0} p_{ij} \cdot v_j$

It follows from the results of Problem 3, that all the values v_i, $i=1,\dots,m$, in this iteration will converge to the average $\bar{v} = \frac{1}{m}\sum_{i=1}^{m} v_i$.

You can find more about these techniques by googling the terms: "distributed averaging" and "averaging consensus".

Problem 4:

I shall provide some discussion on this problem in class.