ISYE 7201: Production & Service Systems
Spring 2018
Instructor: Spyros Reveliotis
1st Midterm Exam (Take Home)
Release Date: February 16, 2018
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Name: SOLUTIONS

While taking this exam, you are expected to observe the Georgia Tech Honor Code. In particular, no collaboration or other interaction among yourselves is allowed while taking the exam.

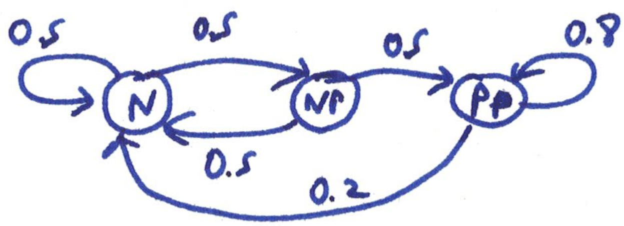
You can send me your responses as a pdf file attached to an email. This pdf file can be a scan of a hand-written document, but, please, write your answers very clearly and thoroughly. Also, report any external sources (other than your textbook and the text by Cassandras and Lafortune that is mentioned in the provided homeworks) that your referred to while preparing the solutions.

Finally, Homeworks 1 and 2 posted at the course website, as well as some of the past midterms posted at that website, can be provide useful complementary material (and experience) to the in-class lectures while working on the exam problems.

Problem 1 (20 points): Consider a sequence of trials with a binary outcome, and with these outcomes presenting some correlation. In particular, if the outcomes of the last two trials were positive, the next trial will have a positive outcome with probability 0.8. Otherwise, the outcome of the next trial will be positive with probability 0.5. Furthermore, each trial costs \$10, and a positive outcome generates a revenue of \$20 while a negative outcome generates zero revenue.

Use the above information in order to compute the (long term) average profit per trial.

The behavior of this sequence of trials can be modeled through the following DTMC:



The interpretation of the various states in the above state transition diagram (STD) are as follows:

N: last outcome was negative

NP: last two outcomes were negative and positive (in this sequence)

PP: last two outcomes were positive

This MC is irreducible, finite-state (and therefore positive recurrent) and aperiodic (see the self-loops at states N and PP). Therefore, it has a limiting distribution.

<u>Remark</u>: It is also interesting to notice how this DTMC has been defined by collecting (tracing) information from <u>a number of past epochs.</u>

Next, we compute the limiting distribution of the previous MC. This can be done by solving the following system of equations:

$$\begin{cases} (\pi_N, \pi_{NP}, \pi_{PP}) = (\pi_N \; \pi_{NP} \; \pi_{PP}) \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0.2 & 0 & 0.8 \end{bmatrix} \\ \sum_i \pi_i = 1. \end{cases}$$

We get

$$\pi_N = 4/11 \; ; \quad \pi_{NP} = 2/11 \; ; \quad \pi_{PP} = 5/11$$

We also compute the expected profit from a single trial at each of the three states as follows:

$$N: \quad 0.5 \times 20 - 10 = 0$$

$$NP: \quad 0.5 \times 20 - 10 = 0$$

$$PP: \quad 0.8 \times 20 - 10 = 6.$$

Then, the (long-term) expected profit per trial is equal to

$$\pi_N \times 0 + \pi_{NP} \times 0 + \pi_{PP} \times 6 = \frac{5}{11} \times 6 = 30/11.$$

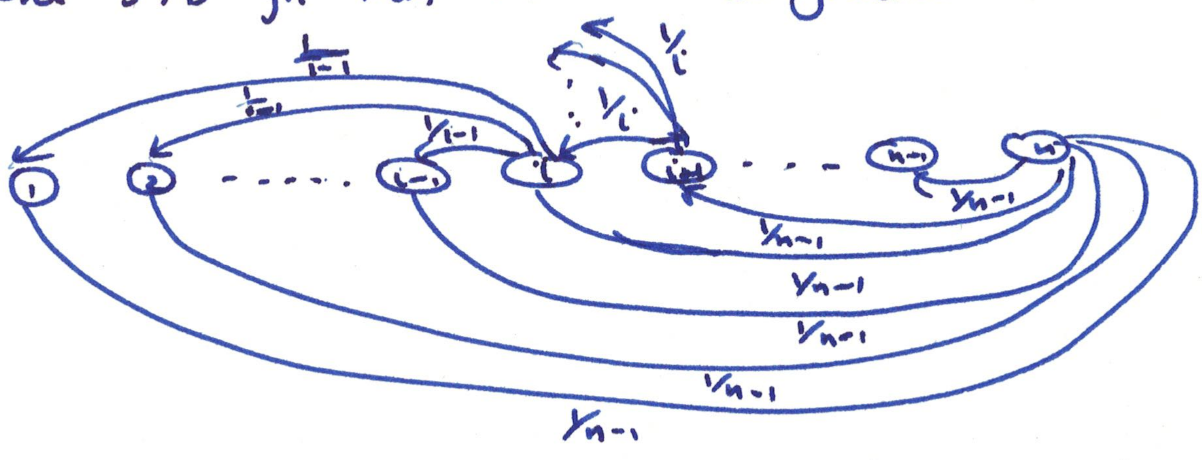
Problem 2 (30 points): Consider a discrete-time Markov chain with state space $X = \{1, \ldots, n\}$, and its one-step transitions being governed by the following law: When the process is in some state $i \in X$, with $i \neq 1$, it transitions with equal probability to any state $j \in \{1, \ldots, i-1\}$. On the other hand, state 1 is an absorbing state (i.e., $p_{11} = 1$). Suppose that the process is started at state n, and let T denote the number of periods until the process reaches its absorbing state 1.

i. (20 pts) Show that $E[T] = \sum_{i=1}^{n-1} \frac{1}{i}$.

ii. (10 pts) Show that $Var[T] = \sum_{i=1}^{n-1} \frac{1}{i}(1 - \frac{1}{i})$.

Hint: It might be useful, especially for part (ii), to represent r.v. T as $T = \sum_{i=1}^{n-1} I_i$, where I_i is a binary random variable indicating whether the process visited state i or not.

The STD for this DTMC is as follows:



Combining the provided hint and the required results of items (i) and (ii), it seems that each r.v. I_i in the expression $T = \sum_{i=1}^{n-1} I_i$ is a Bernoulli trial with corresp. "success" probability $1/i$.

Also, in order to get the results of (i) and (ii) from the above guess, we shall need to establish mutual independence of the I_i's. Next, we proceed to establish these two results.

To show that $P\{I_i = 1\} = E[I_i] = 1/i$, we shall use induction with the "base case" being $i = n-1$.

Indeed, it is clear that the only way node $n-1$ can be visited, is through the one-step transition from n. And the probability of this transition is $1/n-1$.

Next, assuming that the sought result holds for nodes ~~$1/i+1$, $1/i+2$, ...~~ $(i+1), (i+2), -, (n-1)$, we shall show that it also holds for node i. We have:

$$P\{I_i = 1\} = P\{I_i = 1 \wedge I_{i+1} = 1\} + P\{I_i = 1 \wedge I_{i+1} = \emptyset\} \quad (1)$$

But
$$P\{I_i = 1 \wedge I_{i+1} = 1\} = P\{I_i = 1 \mid I_{i+1} = 1\} P\{I_{i+1} = 1\} =$$

$$= \frac{1}{i}\,\frac{1}{i+1} \quad (2)$$ (the first factor in this product results from the structure of our DTMC, and the second from the induction hypothesis).

Also $$P\{I_i = 1 \wedge I_{i+1} = \emptyset\} = P\{I_{i+1} = 1\} = 1/i+1 \quad (3)$$

To see the validity of the first part of the above equation, notice that the sample paths that materialize the event $I_i = 1 \wedge I_{i+1} = \emptyset$ are exactly the same paths that materialize the event $I_{i+1} = 1$ with the last transition in those paths shifted from node $i+1$ to node i, and furthermore, this shift does not alter the realization prob. of the corresponding path.

From (1), (2) and (3), we have:

$$P\{I_i = 1\} = \frac{1}{i}\,\frac{1}{i+1} + \frac{1}{i+1} = \frac{1}{i+1}\left[\frac{1}{i} + 1\right] = \frac{1}{i} \quad (4)$$

To show the mutual independence of I_i's, it suffices to show that $\forall\, i,j$ with $j>i$,

$$P\{I_i=1 \mid I_j=1\} = P\{I_i=1\} = 1/i \qquad (5)$$

This can be argued as follows:

Consider any sample path of the considered MC that has reached node j. Then, it is easy to see that from that point on, the process will evolve as a MC with the same structure with the original one, but with the initial node being node j. But then, the result of equation (5) is immediately obtained from the result of Eq. (4).

Now, we can return to the results of parts (i) and (ii):

For part (i):

$$E[T] = E\left[\sum_{i=1}^{n-1} I_i\right] = \sum_{i=1}^{n-1} E[I_i] = \sum_{i=1}^{n-1} P[I_i=1] =$$

$$= \sum_{i=1}^{n-1} 1/i$$

For part (ii)

$$Var[T] = Var\left[\sum_{i=1}^{n-1} I_i\right] \overset{\text{independence}}{=} \sum_{i=1}^{n-1} Var[I_i] =$$ (each I_i is a Bernoulli trial with $p_i = 1/i$)

$$= \sum_{i=1}^{n-1} \frac{1}{i}\left(1-1/i\right)$$

A more standard approach for part (i) is to consider the r.v.'s T_i, $i=1,\dots,n$, where each T_i denotes the absorption time to state i assuming that the process is started at state i, and develop a system of <u>linear difference equations</u> for their expected values $E[T_i]$.

More specifically, conditioning upon the first step out of each state i, we get:

$$E[T_1] = 0$$

$$\forall i \neq 1: \quad E[T_i] = 1 + \frac{1}{i-1}\sum_{j=1}^{i-1} E[T_j]$$

This system of difference equations can be solved using the corresponding methodology that is provided in Appendix D of your textbook.

It is also useful to try to understand the structure of the solution by trying to compute a few elements of the corresponding sequence. In this case, we get:

$$E[T_2] = 1 + 0 = 1 \quad ; \quad E[T_3] = 1 + \frac{1}{2}E[T_2] = 1 + 1/2$$

which agree with the formula that is provided by the problem.

Next, we shall prove the validity of this formula by induction, using the above computation as the "base" case.

We have:

$$E[T_i] = 1 + \frac{1}{i-1}\left[\sum_{j=1}^{i-2} E[T_j] + E[T_{i-1}]\right] =$$

$$= 1 + \frac{1}{i-1}\Big[\left(E[T_{i-1}]-1\right)\times(i-2) + E[T_{i-1}]\Big] =$$

$$= 1 + \frac{1}{i-1}\Big[(i-1)E[T_{i-1}] + (i-2)\Big] = E[T_{i-1}] + 1/(i-1) = \sum_{j=1}^{i-2} 1/j + 1/(i-1)$$

from inductive hypothesis

Problem 3 (30 points): Consider a local gas station where customers arrive according to a Poisson process with rate $\lambda = 30$ customers per hour. The probability that the customer will pump a particular type of gas is $p = 0.4$, and if the customer chooses this gas type, she will pump a quantity that is uniformly distributed in the interval of 5 and 15 gallons.

Assuming no experienced stock outs, compute the mean and the st. deviation of the amount of the considered gas type that will be sold by this gas station over an interval of 6 hours.

Hint: The sought quantity constitutes a *"compound"* random variable. For such random variables, the mean and the variance can be effectively computed by using the notion of 'conditional expectation'.

Let N = # of arriving customers that pump the considered gas type over the considered time interval.

Then, $N \sim \text{Poisson}(\lambda p t)$

where: $\lambda = 30\ \text{hr}^{-1}$; $p = 0.4$; $t = 6\text{hr}$

Also, let X = total amount of the considered gas type pumped over the considered time interval

Then,
$$X = \sum_{i=1}^{N} X_i$$
where r.v. X_i denotes the amount of gas pumped by the i-th customer who pumped this gas over the considered period.

Obviously $X_i \sim U[5, 15]$ and they are mutually independent.

We shall get $E[X]$ and $Var[X]$ using the concept of conditional expectation. We have:

$$E[X] = E_N\left[E[X \mid N]\right] = E_N\left[E\left[\sum_{i=1}^{N} X_i \mid N\right]\right] =$$

$$= E_N\left[N \cdot E[X_1]\right] = E[N]\,E[X_1] \Rightarrow$$

$$\Rightarrow E[X] = (0.4 \times 30 \times 6)\,\frac{5+15}{2} = 720 \text{ gallons.}$$

For $Var[X]$, we have:

$$Var[X] = E[X^2] - E^2[X]$$

But $E[X^2] = E_N\left[E[X^2 \mid N]\right] = E_N\left[E\left[\left(\sum_{i=1}^{N} X_i\right)^2 \mid N\right]\right]$

(since X_i are iid)

$$= E_N\left[N \cdot E[X_1^2] + N(N-1)\,E^2[X_1]\right] =$$

$$= E[N]\,E[X_1^2] + E[N^2]\,E^2[X_1] - E[N]\,E^2[X_1]$$

Hence, $Var[X] = E[N]\,E[X_1^2] + E[N^2]\,E^2[X_1] - E[N]\,E^2[X_1] -$

$$- E^2[N]\,E^2[X_1] =$$

$$= E[N]\left(E[X_1^2] - E^2[X_1]\right) + E^2[X_1]\left(E[N^2] - E^2[N]\right) =$$

$$= E[N]\,Var[X_1] + E^2[X_1]\,Var[N]$$

In the considered case, $E[N] = Var[N] = \lambda pt$ and therefore

$$Var[X] = (\lambda pt)\left(Var[X_1] + E^2[X_1]\right) = \lambda pt\,E[X_1^2].$$

But $E[X_1^2] = \frac{1}{10}\int_5^{15} x^2\,dx = \frac{1}{30}(15^3 - 5^3) = 108.\overline{3}$

Putting everything together, we get $Var[X] = 0.4 \times 30 \times 6 \times 108.\overline{3} = 7800$ gallons^2

Problem 4 (20 points): Consider n independent Poisson processes, with the i-th process having rate λ_i. Derive an expression for the expected time until an event has occurred in all n processes.

Instead of writing up a solution for this problem, I decided to use the solution that I received by one of your fellow students; please, see the attached page. This is a very nice and succinct presentation of the corresponding results!

I want also to make the following two remarks:

Remark 1: The first formula that is derived for $E[T]$ (i.e., $E[T] = \int_0^\infty (1 - F_T(t))\,dt \equiv \int_0^\infty \bar{F}_T(t)\,dt$) is an identity that holds for all continuous r.v.'s with positive support. A similar result holds for positive discrete r.v.'s where the ~~sum~~ integral is replaced by summation.

Remark 2: This problem has a very strong affinity to the "coupon collecting" problem that is treated as problem (e) in the Homework #3 that is posted at the course website.

Problem 4 20

Let T be the time until an event has occurred in every process. For $t \in \mathbb{R}^+$

$$\begin{aligned}
\mathrm{P}[T \le t] &= \mathrm{P}[\text{an event has occured for every process before } T] \\
&= \prod_{i=1}^{n} \mathrm{P}[\text{an event has occured for process } i \text{ before } T] \\
&= \prod_{i=1}^{n} \mathrm{P}[N_i(t) > 0] \\
&= \prod_{i=1}^{n} \left(1 - e^{-\lambda_i t}\right).
\end{aligned}$$

Here $N_i(t)$ denotes the Poisson process with rate λ_i.

Then $\mathbb{E}[T] = \int_0^\infty t f_T(t) dt = \int_0^\infty \left[\int_0^t 1\, du\right] f_T(t) dt = \int_0^\infty \int_t^\infty f_T(u)\, du\, dt = \int_0^\infty (1 - F_T(t))\, dt,$

where $F_T(\cdot)$ is the cumulative distribution function of T. We thus have

$$\begin{aligned}
\mathbb{E}[T] &= \int_0^\infty \left[1 - \prod_{i=1}^{n} \left(1 - e^{-\lambda_i t}\right)\right] dt \\
&= - \sum_{\Omega \neq \emptyset; \Omega \subset [n]} (-1)^{|\Omega|} \int_0^\infty \exp\left\{\sum_{\omega \in \Omega} -\lambda_\omega t\right\} dt \\
&= - \sum_{\Omega \neq \emptyset; \Omega \subset [n]} (-1)^{|\Omega|} \int_0^\infty \exp\left\{-\sum_{\omega \in \Omega} \lambda_\omega t\right\} dt \\
&= - \sum_{\Omega \neq \emptyset; \Omega \subset [n]} (-1)^{|\Omega|} \frac{1}{\sum_{\omega \in \Omega} \lambda_\omega} \\
&= - \sum_{i=1}^{n} \sum_{\Omega \subset [n]; |\Omega| = i} (-1)^i \frac{1}{\sum_{\omega \in \Omega} \lambda_\omega} \\
&= \sum_{i=1}^{n} \lambda_i^{-1} - \sum_{\{i,j\} \subset [n]} \left(\lambda_i + \lambda_j\right)^{-1} + \sum_{\{i,j,k\} \subset [n]} \left(\lambda_i + \lambda_j + \lambda_k\right)^{-1} - \cdots + (-1)^{n+1} \left(\sum_{i=1}^{n} \lambda_i\right)^{-1}.
\end{aligned}$$

The first "=" comes from the expansion of the product and the interchange of the summation and integration. Here $[n]$ denotes the set $\{1, \dots, n\}$.