ISYE 7201: Production & Service Systems
Spring 2017
Instructor: Spyros Reveliotis
Midterm Exam II (Take-Home)
Due Date: April 19, 2017

Name:

Problem 1 (30 points): Consider a population where each individual is assumed to give birth at an exponential rate λ and to die at an exponential rate μ. In addition, there is an exponential rate of increase θ due to immigration. However, immigration is not allowed when the population size is N or higher.

i. (10 pts) Model the dynamics of this population as a continuous-time Markov chain.

ii. (10 pts) What is the stability condition for this Markov chain?

iii. (10 pts) If $N = 3$, $\lambda = \theta = 1$, and $\mu = 3$, determine the proportion of time that immigration is restricted.

Problem 2 (30 points): Consider N machines operating independently from each other. Machine i operates for an exponential time with rate λ_i, and then it fails. Its repair time is exponential with rate μ_i.

i. (10 pts) Define a continuous-time Markov chain to describe jointly the evolution of the operational status of these N machines over time.

ii. (10 pts) Assume that all machines are operational at time 0, and compute the probability that all machines are operational at some given time t.

iii. (10 pts) Explain that the CTMC defined at item (i) above has a limiting distribution, and compute the "steady-state" probability that there is at least one operational machine.

Hint: It is important to notice that the N machines operate independently from each other. Then, try to refer the operation of each machine to some CTMC model studied in class.

Problem 3 (20 points): Consider a network of three stations with a single server at each station. Customers arrive at stations 1,2,3 according to Poisson processes having respective rates 5, 10 and 15. The service times at the three stations are exponential with respective rates 10, 50 and 100. A customer completing service at station 1 is equally likely to (i) go to station 2, (ii) go to station 3, or (iii) leave the system. A customer completing service at station 2 always goes to station 3. Finally, a customer completing service at station 3 is equally likely to either go to station 2 or leave the system.

i. (10 pts) Argue that the above network has a limiting distribution and compute the average number of customers in it in steady state.

ii. (10 pts) What is the average time a customer spends in the system?

Problem 4 (20 points): Consider a stable M/M/1 queueing station with arrival rate λ and processing rate μ. Furthermore, assume that the station starts empty, and let T_n denote the time that it gets n customers in it for the first time. Show that

$$E[T_n] = \frac{1}{\lambda - \mu} \cdot \left[n - \frac{\mu}{\lambda} \cdot \frac{1 - (\mu/\lambda)^n}{1 - \mu/\lambda} \right]$$

Hint: Let U_i denote the random time that it takes to get from state i to state $i+1$, for $i \geq 0$, and notice that $E[T_n] = \sum_{i=0}^{n-1} E[U_i]$. Then, develop a recursion to compute the $E[U_i]$, for $i \geq 0$, in terms of the parameters λ and μ.

ISYE 7201

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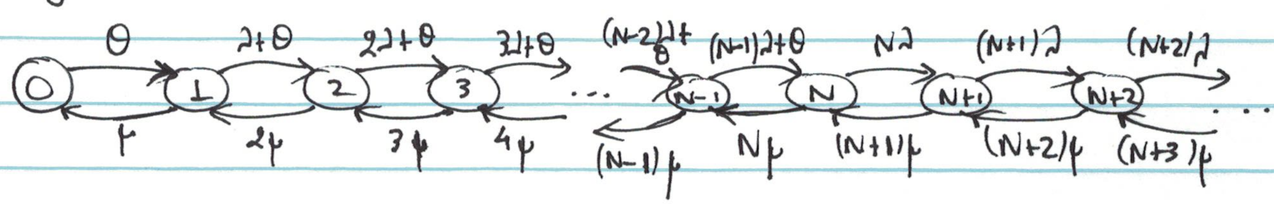
Midterm II Solutions

Problem 1:

(i) The state of the considered CTMC is the population size at time t, and the corresponding state transition diagram is as follows:



(ii) Let P_i denote the steady-state probability for state i. Then, the corresponding balance equations are as follows:

For the first N states (i.e., for states $\emptyset$ to $N-1$) we have:

State $\emptyset$: $\theta P_0 = \mu P_1 \Rightarrow P_1 = \frac{\theta}{\mu} P_0$

State 1: $(\lambda+\theta+\mu)P_1 = \theta P_0 + 2\mu P_2 \Rightarrow P_2 = \left[\frac{\theta}{\mu}(\lambda+\theta+\mu) - \theta\right]\frac{P_0}{2\mu} =$

$$= \frac{\theta(\lambda+\theta)}{2\mu^2} P_0$$

State 2: $(2\lambda+\theta+2\mu)P_2 = (\lambda+\theta)P_1 + 3\mu P_3 \Rightarrow$

$$\Rightarrow P_3 = \frac{P_0}{3\mu}\left[\frac{\theta(\lambda+\theta)}{2\mu^2}(2\lambda+\theta+2\mu) - (\lambda+\theta)\frac{\theta}{\mu}\right] = \frac{P_0}{\underbrace{6\mu^3}_{3!}}\theta(\lambda+\theta)(2\lambda+\theta)$$

And working through induction, we can show that

$\forall i \in \{0,\dots,N\}$, $$P_i = \frac{\prod_{j=0}^{i-1}(j\lambda+\theta)}{i!\,\mu^i} P_0 \qquad (1)$$

For $i \geq N+1$, we use the balance equations

$$P_i\,(i\mu) = P_{i-1}\,[(i-1)\lambda] \implies P_i = \frac{(i-1)\lambda}{i\mu}\,P_{i-1}$$

Hence, for $i \geq N+1$,

$$P_i = \frac{(i-1)(i-2)\cdots N}{i(i-1)\cdots(N+1)}\left(\frac{\lambda}{\mu}\right)^{i-N} P_N = \frac{N}{i}\left(\frac{\lambda}{\mu}\right)^{i-N} P_N \quad (2)$$

For stability, we need the normalizing equation $\sum_{i=0}^{\infty} P_i = 1$ to have a well-defined, positive solution w.r.t. the unknown variable P_0. From Eqs (1) and (2), this last equation can be written as follows:

$$(3) \quad P_0\left\{\sum_{i=0}^{N-1}\frac{\prod_{j=0}^{i-1}(\lambda_j+\theta)}{i!\,\mu^i} + \frac{\prod_{j=0}^{N-1}(\lambda_j+\theta)}{N!\,\mu^N}\cdot N\sum_{i=N}^{\infty}\frac{1}{i}\left(\frac{\lambda}{\mu}\right)^{i-N}\right\} = 1$$

Eq. (3) will have a unique positive solution for P_0 if and only if

$$\sum_{i=N}^{\infty}\frac{1}{i}\left(\frac{\lambda}{\mu}\right)^{i-N} = \sum_{i=0}^{\infty}\frac{1}{N+i}\left(\frac{\lambda}{\mu}\right)^{i} < \infty \quad (4)$$

From D'Alembert's convergence criterion for infinite series, the condition of Eq. (4) will be satisfied iff $\exists\, r<1$ s.t.

$$\frac{\frac{1}{N+1+i}\left(\frac{\lambda}{\mu}\right)^{i+1}}{\frac{1}{N+i}\left(\frac{\lambda}{\mu}\right)^{i}} = \frac{N+i}{N+1+i}\,\frac{\lambda}{\mu} \leq r, \quad \forall i$$

This last condition will be true if and only if $\lambda/\mu < 1$ (5)

(iii) Clearly, the proportion of time that immigration is restrained is

$$\sum_{i=3}^{\infty} P_i = \sum_{i=3}^{\infty} \frac{3}{i}\left(\frac{\lambda}{\mu}\right)^{i-3} P_3 \qquad (6)$$

From $\sum_{i=0}^{\infty} P_i = 1$ we have:

$$P_0 + \frac{\theta}{\mu}P_0 + \frac{\theta(\lambda+\theta)}{2\mu^2}P_0 + \frac{\theta(\lambda+\theta)(\lambda+2\theta)}{6\mu^3}P_0 +$$
$$+ \sum_{i=3}^{\infty} \frac{3}{i}\left(\frac{\lambda}{\mu}\right)^{i-3} \frac{\theta(\lambda+\theta)(2\lambda+\theta)(3\lambda+\theta)}{24\mu^4} P_0 = 1 \qquad (7)$$

Eq. (7), combined with the fact that for $\lambda/\mu < 1$

$$\sum_{i=1}^{\infty} \frac{1}{i}\left(\frac{\lambda}{\mu}\right)^{i} = \ln\left(\frac{\mu}{\mu-\lambda}\right),$$ can be used to obtain P_0.

Then, P_3 can be obtained from Eq. (1) as

$$P_3 = \frac{\theta(\lambda+\theta)(2\lambda+\theta)(3\lambda+\theta)}{24\mu^4} P_0$$

and plugging the corresponding result in Eq. (6), we obtain

$$\sum_{i=3}^{\infty} P_i = \frac{\left\{\frac{1}{2\lambda^3}\left[\ln\left(\frac{\mu}{\mu-\lambda}\right) - \frac{\lambda}{\mu} - \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2\right]\theta(\lambda+\theta)(2\lambda+\theta)\right\}}{1 + \frac{\theta}{\mu} + \frac{\theta(\lambda+\theta)}{2\mu^2} + \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{2\lambda^3}\left[\ln\left(\frac{\mu}{\mu-\lambda}\right) - \frac{\lambda}{\mu} - \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2\right]}$$

Problem 2:

(i) The state of this CT-MC are all the N-dimensional binary vectors where a value of '1' for the i-th component implies that the i-th machine is up.

From any given state $x = (b_1, b_2, \ldots, b_N)$, the process can transition only to a state x' that differs from x at one component only. If this component goes from 1 to $\emptyset$, the corresponding rate is λ_i; in the opposite case, the rate is μ_i.

(ii) Since the N machines operate independently, the required probability is $\prod_{i=1}^{N} P[X_i(t) = 1 \mid X_i(\emptyset) = 1]$

But each machine operates according to the following CT-MC

$$\begin{tikzcd} 1 \arrow[r, bend left, "\lambda_i"] & 0 \arrow[l, bend left, "\mu_i"] \end{tikzcd} \qquad i = 1, \ldots, N$$

and the transition probabilities for this MC were computed in class as follows:

$$P_{11}(t) = P[X_i(t) = 1 \mid X_i(\emptyset) = 1] = \frac{\mu_i}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i} e^{-(\lambda_i + \mu_i)t}$$

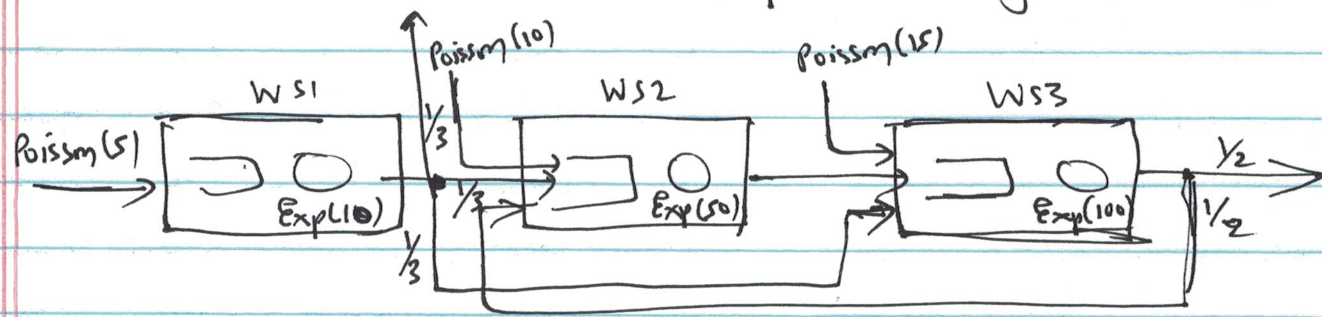
(iii) The considered CT-MC has a finite state space, and this state space constitutes a single communicating class. Hence, the embedded DT-MC is positive recurrent. Furthermore, the expected sojourn time at any given state is finite. Therefore this CT-MC satisfies all the requirements of the corresponding theorem discussed in class for the existence of a limiting distribution.

The probability that $\exists$ at least one operational machine is

$$1 - \text{Prob(all machines down)} = 1 - \prod_{i=1}^{N} \lim_{t \to \infty} P_{10}(t) = 1 - \prod_{i=1}^{N} \lambda_i / (\lambda_i + \mu_i)$$

Problem 3:

The considered QN can be depicted as follows:



(i) First we compute the total arrival rates to the three stations, $\lambda_i, i=1,2,3$. We have:

$$\begin{cases} \lambda_1 = 5 \\ \lambda_2 = 10 + \frac{1}{3}\lambda_1 + \frac{1}{2}\lambda_3 \\ \lambda_3 = 15 + \frac{1}{3}\lambda_1 + \lambda_2 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 5 \\ \lambda_2 = 40 \\ \lambda_3 = \frac{170}{3} \simeq 56.67 \end{cases}$$

For stability, we need $\rho_i = \lambda_i / \mu_i$, $\forall i$, which is easily checked to be true.

Then,

$$L_1 = \frac{\rho_1}{1-\rho_1} = \frac{5/10}{1-5/10} = 1$$

$$L_2 = \frac{\rho_2}{1-\rho_2} = \frac{40/50}{1-40/50} = 4$$

$$L_3 = \frac{\rho_3}{1-\rho_3} = \frac{170/(3\times100)}{1-170/(3\times100)} = \frac{17}{13}$$

(iii) From Little's law we have:

$$CT = \frac{L_1 + L_2 + L_3}{\alpha_1 + \alpha_2 + \alpha_3} = \frac{1+4+17/13}{5+10+15} \simeq 0.21$$

Problem 4:

Using the notation that is introduced in the problem description, we have:

$$E[U_0] = 1/\lambda$$

$$\forall i \geq 1,\quad E[U_i] = \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\left[E[U_{i-1}] + E[U_i]\right] \Rightarrow$$

$$\Rightarrow E[U_i] = \frac{\lambda+\mu}{\lambda}\left[\frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}E[U_{i-1}]\right] = \frac{1}{\lambda} + \frac{\mu}{\lambda}E[U_{i-1}]$$

$$= \frac{1}{\lambda}\left[1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \cdots + \left(\frac{\mu}{\lambda}\right)^{i-1}\right] + \left(\frac{\mu}{\lambda}\right)^i E[U_0] =$$

$$= \frac{1}{\lambda}\sum_{j=0}^{i}\left(\frac{\mu}{\lambda}\right)^j = \frac{1}{\lambda}\,\frac{1-(\mu/\lambda)^{i+1}}{1-(\mu/\lambda)} = \frac{1-(\mu/\lambda)^{i+1}}{\lambda-\mu}$$

Then,

$$E[T_n] = \sum_{i=0}^{n-1} E[U_i] = \frac{1}{\lambda-\mu}\sum_{i=0}^{n-1}\left[1-(\mu/\lambda)^{i+1}\right] =$$

$$= \frac{1}{\lambda-\mu}\left[n - \frac{\mu}{\lambda}\sum_{i=0}^{n-1}\left(\frac{\mu}{\lambda}\right)^i\right] = \frac{1}{\lambda-\mu}\left[n - \frac{\mu}{\lambda}\,\frac{1-(\mu/\lambda)^n}{1-(\mu/\lambda)}\right]$$