ISYE 7201: Production & Service Systems
Spring 2017
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Midterm Exam
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Name: SOLUTIONS

Problem 1 (20 points): Consider a stock of a certain commodity that is managed according to the, so called, (s, S) ordering policy: $S > s$, and if at the end of the day the stock is at level $x \geq s$, then no replenishment order is placed; otherwise, there is a replenishment order of $S - x$. This order is delivered overnight, and it is available when the store opens next morning.

Daily demands are independent and equal to j units with probability p_j. Demand that cannot be met immediately from stock is lost.

Let X_n denote the inventory level at the end of the n-th day. Argue that $\{X_n, n \geq 1\}$ is a Markov chain, define the corresponding state space, and compute the corresponding one-step transition probability matrix.

Clearly, the state space of this stochastic process is $\mathcal{X} = \{0, 1, 2, \ldots, S\}$

Furthermore, given X_n, X_{n+1} will depend upon X_n, the order placed at the end of day n (which is determined completely by X_n) and the demand in day $n+1$. So $\{X_n, n \geq 1\}$ is a DTMC.

The one-step transition probabilities can be defined as follows:

$\forall i \in \{0, 1, 2, \ldots, s-1\}$,

$$P_{ij} = \begin{cases} p_{S-j} & j = 1, \ldots, S \\ \sum_{k=S}^{\infty} p_k & \end{cases}$$

$\forall i \in \{s, s+1, \ldots, S\}$,

$$P_{ij} = \begin{cases} \emptyset & j = i+1, \ldots, S \\ p_{i-j} & j = 1, 2, \ldots, i \\ \sum_{k=i}^{\infty} p_k & \end{cases}$$

Problem 2 (30 points):

i. (10 pts) Consider a DTMC $\{X_n, n \geq 0\}$, and for states i, j, k with $k \neq j$ let:

$$p_{ij/k}^{(n)} = Prob\{X_n = j, X_l \neq k, l = 1, \ldots, n-1 | X_0 = i\}.$$

Explain in words what $p_{ij/k}^{(n)}$ represents.

ii. (10 pts) Prove that, for $i \neq j$, $p_{ij}^{(n)} = \sum_{k=0}^{n} p_{ii}^{(k)} p_{ij/i}^{(n-k)}$

iii. (10 pts) Prove that

$$Prob\{X_k = i_k | X_j = i_j \text{ for all } j \neq k\} = Prob\{X_k = i_k | X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}\}.$$

(i) This is the probability of going from i to j in n steps without going through k.

(ii) Consider the set of sample paths that take the process from i to j in n steps. This set can be partitioned on the basis of the last visit to state i. Let the period of this last visit be denoted by k. Then, for any given k, the corresponding probability is

$$p_{ii}^{(k)} \; p_{ij/i}^{(n-k)}$$

and the result is obtained from the aforestated fact that k partitions the entire set of sample paths of interest.

(iii) $P\{X_k = i_k \mid X_j = i_j \ \ \forall j \neq k\} =$

$$= \frac{P\{X_k = i_k;\ X_j = i_j,\ \forall j: j>k \mid X_j = i_j,\ \forall j: j<k\}}{P\{X_j = i_j,\ \forall j: j>k \mid X_j = i_j,\ \forall j: j<k\}} =$$

$$= \frac{P\{X_k = i_k;\ X_j = i_j,\ \forall j: j>k \mid X_{k-1} = i_{k-1}\}}{P\{X_j = i_j,\ \forall j: j>k \mid X_{k-1} = i_{k-1}\}} =$$

$$= \frac{p_{i_{k-1} i_k}\, p_{i_k i_{k+1}}\, p_{i_{k+1} i_{k+2}} \cdots}{p^{(2)}_{i_{k-1} i_{k+1}}\, p_{i_{k+1} i_{k+2}} \cdots} = \frac{p_{i_{k-1} i_k}\, p_{i_k i_{k+1}}}{p^{(2)}_{i_{k-1} i_{k+1}}} =$$

$$= P\{X_k = i_k \mid X_{k-1} = i_{k-1},\ X_{k+1} = i_{k+1}\}$$

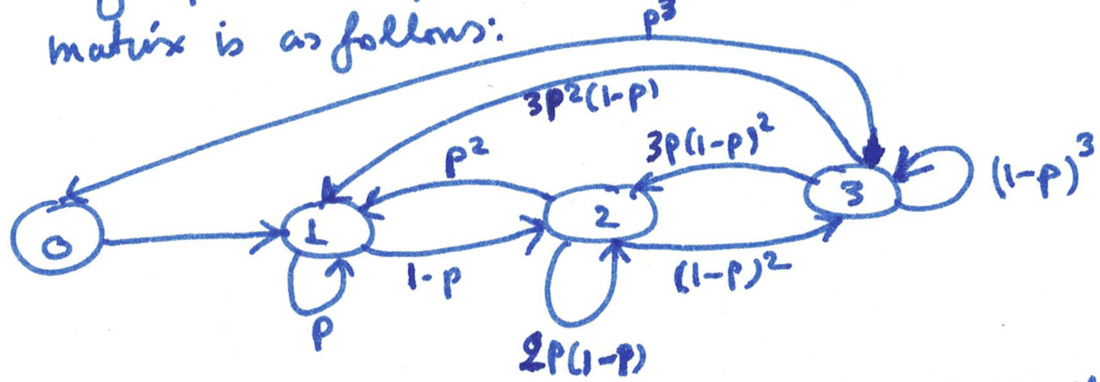
Problem 3 (30 points): A local shop floor employs three machines that during their daily operation will fail with probability p, independently from each other. Machines that are down at the end of a day will be repaired by a single technician at the rate of one machine per day. Let X_n denote the number of machines that are operational at the beginning of the n-th day.

i. (10 pts) Show that $\{X_n, n \geq 0\}$ is a DTMC and define the corresponding one-step transition probability matrix.

ii. (10 pts) Argue that this Markov chain has a limiting distribution $\{\pi_i,\ i = 0, 1, 2, 3\}$.

iii. (10 pts) If the outage of any single machine for a day incurs a loss of a dollars, characterize the (long run) average loss per day for the considered shop floor due to the experienced outages.

(i) A graphical representation of this trans. prob. matrix is as follows:



(ii) This Markov chain is finite-state, irreducible, and therefore positive recurrent.
The self-loops appearing in the chain also imply that it is aperiodic.

(iii) Let $\{\pi_i, i = 0, 1, 2, 3\}$ denote the steady-state prob. of this chain. Then, the considered quantity can be expressed by $a(3\pi_0 + 2\pi_1 + \pi_2)$

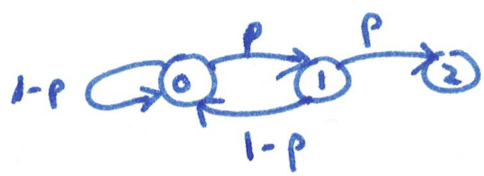
Problem 4 (20 points): Consider a Poisson process with rate $\lambda = 1$ event per time unit. Compute the expected time until the occurrence of two consecutive events with respective inter-event times of at least two time units.

The probability that an inter-event time ~~exceeds~~ is at least two time units is

$$p = P\{X \geq 2\} = e^{-2} \simeq 0.1353.$$

Furthermore, every inter-event period has an average duration $\tau = 1/\lambda = 1$ time unit.

But then, the required expected time can be computed as the time-to-absorption of the following DTMC:



Let f_i denote the expected time to absorption in state 2 when the process is started at state $i = 1, 2$. Then, we have:

$$\begin{cases} f_0 = 1 + p f_1 + (1-p) f_0 \\ f_1 = 1 + (1-p) f_0 \end{cases} \Rightarrow f_0 = 1 + p + p(1-p) f_0 + (1-p) f_0$$

$$\Rightarrow (\cancel{1}\cancel{-p} + p^2 \cancel{-1}\cancel{+p}) f_0 = 1 + p \Rightarrow$$

$$\Rightarrow f_0 = \frac{1+p}{p^2} = \frac{1+e^{-2}}{e^{-4}} \simeq 62.02$$