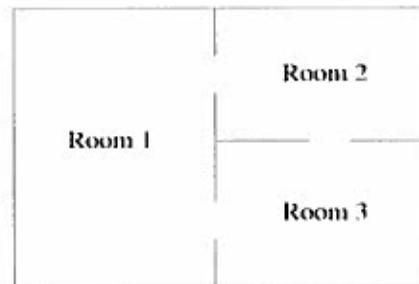


ISYE 7201: Production & Service Systems
Spring 2016
Instructor: Spyros Reveliotis
Midterm Exam
March 9, 2016

Name: SOLUTIONS

Problem 1 (30 points): Consider a cat and a mouse that move around a house with the following layout:



Each animal stays at a visited room for a random time T_x , $x \in \{c, m\}$, with mean $\tau_x < \infty$, and then it moves to one of the other rooms with equal probability.

- i. (10 pts) Argue that no matter what are the initial locations at time 0, the animals will find themselves in the same room in finite time.
- ii. (10 pts) Assuming that the sojourn times spent at the different rooms by each animal are exponentially distributed, develop an analytical method for computing the expected time that it will take for the first occurrence of the event described in item #1 above, assuming that the animals start in different rooms. Please, provide as many details as you can.
- iii. (5 pts) Use your work in parts (i) and (ii) above to investigate how the expected time that is computed in part (ii) is impacted by the initial placement of the animals in the three rooms (always under the assumption that this placement puts the animals in separate rooms).
- iv. (5 pts) Does your method apply to the case that sojourn times are not exponentially distributed by they are drawn from some other distributions with the same means?

- (i) Consider the two-state stochastic process that checks whether the two animals are in the same room or not, with state 0 indicating that they are in different rooms and state 1 indicating that they are in the same room.

Then, according to the problem description, the stochastic process starts at state 0 at $t=0$ and we are interested in its absorption in state 1. The corresponding dynamics can be modelled as follows:



The notation p_n in the above chart indicates the probability that the process moves from state 0 to state 1 during its n -th transition. Each of these transitions can be a single-animal move or a ~~move~~ combined move by both animals. In the first case it is easily checked that $p_n = 1/2$ while in the second case $p_n = 1/4$. These numbers imply the $E[\text{\# of transitions to absorption}] \leq 4$. And since sojourn times at state 0 are finite, the absorption will take place in finite time.

- (ii) In this case, the stochastic process introduced in step (i) is a CTMC with sojourn time at state 0 exponentially distributed with mean ~~value~~ value $\tau = 1/(\frac{1}{\tau_c} + \frac{1}{\tau_m}) = \tau_c \tau_m / (\tau_c + \tau_m)$. Also, in this case $p_n = 1/2$, $\forall n$ since the probability for a simultaneous transition by both animals is an event of zero measure. Hence, $E[\text{time to absorption}] = \frac{2\tau_c \tau_m}{\tau_c + \tau_m}$

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(iii) It should be clear from all the previous discussion that the problem symmetries imply that the expected time to absorption in a state where both animals are in the same room does not depend on the initial location of the animals (as long as these two locations are different).

(iv) Also, it is clear that the method developed in part (ii) depends on the exponential nature of the animal sojourn times at each visited room. In the case of more general distributions for these sojourn times, the sojourn times for state $\emptyset$ in the stoch. process of step (i) will depend on the residual sojourn time in the current room for the animal that did not move (in case of single-move transitions). This dependency renders the considered stoch. process a Generalized semi-Markov process (GSMP), and the computation of the sought expectation quite difficult.

On the other hand, we can still compute an upper bound for this expectation using the lower bound for p_n or $1/4$ and setting an upper bound for the expected sojourn time at state $\emptyset$ equal to $\max\{\tau_c, \tau_m\}$. Then

$$E[\text{time to absorption}] \leq 4 \max\{\tau_c, \tau_m\}$$

Problem 2 (20 points): Consider a queuing system with m parallel exponential servers with common processing rate μ . Currently, all servers are busy and also there are n customers waiting in the queue. Upon observing this state, the management decides to shut off the admission process, and let the system clear its current workload. Assuming a non-idling policy on the part of the servers, determine the *mean* and the *variance* of the necessary time to clear this workload.

According to the problem description, currently we have $m+n$ customers in the system, and these customers will be processed by m parallel servers, each with proc. time distribution $\text{Exp}(\mu)$.

Hence, letting T denote the total proc. time, we have:

$$E[T] = (n+1) \frac{1}{m\mu} + \frac{1}{(m-1)\mu} + \dots + \frac{1}{\mu} + \frac{1}{\mu}$$

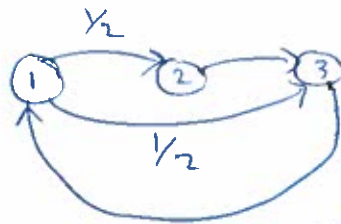
$$\text{Var}[T] = (n+1) \frac{1}{(m\mu)^2} + \frac{1}{(m-1)^2\mu^2} + \dots + \frac{1}{2^2\mu^2} + \frac{1}{\mu^2}$$

Problem 3 (30 points): Consider a 3-state continuous-time Markov chain with the following infinitesimal generator matrix:

$$\begin{bmatrix} -1.0 & 0.5 & 0.5 \\ 0.0 & -0.5 & 0.5 \\ 1.0 & 0.0 & -1.0 \end{bmatrix}$$

- i. (10 pts) Argue that this chain has a limiting probability distribution and compute this distribution.
- ii. (10 pts) Furthermore, suppose that when visiting state i , $i \in \{1, 2, 3\}$, the process collects a reward with rate r_i \$ per time unit spent at that state. These rates are provided by the vector $\mathbf{r} = [10, 5, 3]$. Compute the average reward per unit of time that is collected by the considered process in steady state.
- iii. (10 pts) Finally, suppose that the quantities that are quoted in vector $\mathbf{r}$ in step (ii) above, are not reward rates, but a lump amount of reward that the process receives every time that it gets into the corresponding state i . Compute the average reward per unit of time that is collected by the considered process in steady state under this new situation.

(i) The embedded DTMC is as follows:



So, it is irreducible, finite state and therefore positive recurrent.

Also, the finiteness of the process states guarantees that $E[T_{ii}] < \infty$, $\forall i = 1, 2, 3$

(ii) Let $p = [p_1, p_2, p_3]$ denote the steady-state prob. for the considered CTMC, and Q the provided infinitesimal generator. Then, from $p \cdot Q = 0$, we get:

$$\begin{cases} -p_1 + p_3 = 0 \\ 0.5p_1 - 0.5p_2 = 0 \end{cases} \Rightarrow p_1 = p_2 = p_3 = \frac{1}{3}$$

Finally $E_p[r] = \frac{1}{3} (10 + 5 + 3) = 18/3 = 6$.

(iii) The one-step transition prob. matrix for the embedded DTMC presented in step (i) is as follows:

$$\hat{P} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Letting $\pi = [\pi_1, \pi_2, \pi_3]$ denote the stationary distribution for this process, we have from $\pi \cdot \hat{P} = \pi$:

$$\begin{cases} \pi_3 = \pi_1 \\ \frac{1}{2}\pi_1 = \pi_2 \end{cases} \Rightarrow 2\pi_1 + \frac{1}{2}\pi_1 = 1 \Rightarrow \begin{cases} \pi_1 = \pi_3 = \frac{2}{5} \\ \pi_2 = \frac{1}{5} \end{cases}$$

The above results also imply that a recurrence cycle that starts from state 2 will involve, on average, two visits to state 1 and two visits to state 3 before state 2 is revisited. Hence, the expected reward collected during such a cycle is $r_2 + 2r_1 + 2r_3 = 5 + 20 + 6 = 31$. On the other hand, the expected length of such a cycle is $\frac{1}{0.5} + 2 \cdot 1 + 2 \cdot 1 = 6$. Hence, the expected reward per time unit for this case is $31/6 = 5.1\bar{6}$.

Problem 4 (20 points): In Homework #2 you were called to argue in rather informal / intuitive terms that, when X is an exponentially distributed random variable, $E[X^2 | X > 1] = E[(X + 1)^2]$.

Now, please, provide an analytical proof for the above result.

Using the relevant results developed in class,

$$\begin{aligned}
 E[X^2 | X > 1] &= \int_1^{\infty} x^2 \frac{f(x)}{\overline{F}(1)} dx = \int_1^{\infty} x^2 \frac{2e^{-2x}}{e^{-2}} dx = \\
 &= \int_1^{\infty} x^2 2e^{-2(x-1)} dx \stackrel{y=x-1}{=} \int_0^{\infty} (y+1)^2 2e^{-2y} dy \\
 &= E[(X+1)^2]
 \end{aligned}$$