HW #5 Solutions

(a)

Problem 8.11

Our model is a Jackson network of five nodes. Let's calculate the actual arrival rates λ_i at each node i:

$$\left.\begin{array}{l} \lambda_1 = r_1 + \lambda_4 p_{41} = 1 + \lambda_4 \\ \lambda_2 = \lambda_1 p_{12} + \lambda_5 p_{52} = 0.8\lambda_1 + \lambda_5 \\ \lambda_3 = \lambda_2 p_{23} = \lambda_2 \\ \lambda_4 = \lambda_1 p_{14} + \lambda_3 p_{34} = 0.2\lambda_1 + 0.2\lambda_3 \\ \lambda_5 = \lambda_3 p_{35} = 0.2\lambda_3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \lambda_1 = \frac{5}{3} \\ \lambda_2 = \frac{5}{3} \\ \lambda_3 = \frac{5}{3} \\ \lambda_4 = \frac{2}{3} \\ \lambda_5 = \frac{1}{3} \end{array}\right\} \text{min}^{-1}$$

a) Each node M_i can be treated as an M/M/1 queueing system with arrival rate λ_i and service rate μ_i.

Hence,

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{5/3}{2} = \frac{5}{6}, \quad \rho_2 = \frac{\lambda_2}{\mu_2} = \frac{5/3}{7/4} = \frac{20}{21}$$

$$\rho_3 = \frac{\lambda_3}{\mu_3} = \frac{5/3}{2.2} = \frac{25}{33}, \quad \rho_4 = \frac{\lambda_4}{\mu_4} = \frac{2/3}{1} = 2/3, \quad \rho_5 = \frac{\lambda_5}{\mu_5} = \frac{1/3}{0.5} = \frac{2}{3}$$

b) The throughput of the system is

$$\lambda_3 p_{30} = \frac{5}{3} \times 0.6 = 1$$

This should be expected since all nodes are stable (all $\rho_i < 1$) and the total incoming rate is $r_1 = 1\ \text{min}^{-1}$

c) if X_i is the queue length at node M_i, $i = 1, 2, \dots, 5$ then the average total number of customers in the system, $E[X]$, is

$$E[X] = \sum_{i=1}^{5} E[X_i] = \sum_{i=1}^{5} \frac{\rho_i}{1-\rho_i} = \frac{5/6}{1-5/6} + \frac{20/21}{1-20/21} + \frac{25/33}{1-25/33} + \frac{2/3}{1-2/3} + \frac{2/3}{1-2/3}$$

$$= 5 + 20 + \frac{25}{8} + 2 + 2 = 32.125$$

The arrival rate λ into the system is

$$\lambda = r_1 = 1$$

By Little's law, the average system time $E[S]$ is

$$E[S] = \frac{E[X]}{\lambda} = \frac{32.125}{1} = 32.125$$

d. $P(X_2>3) = 1 - P(X_2=0) - P(X_2=1) - P(X_2=2) - P(X_2=3)$

$$= 1-(1-p_2)-(1-p_2)p_2-(1-p_2)p_2^2-(1-p_2)p_2^3$$

$$= 1-\frac{1}{21}-\frac{1}{21}\frac{20}{21}-\frac{1}{21}\left(\frac{20}{21}\right)^2-\frac{1}{21}\left(\frac{20}{21}\right)^3$$

$$\simeq 0.823$$

Problem 8.14

Following the notation that we used in the discussion of the mean value analysis for closed QNs, let

$W_m(j)$ = Average sojourn time at node j for an m-customer network,

Then, for part (a), we essentially need to compute the quantity

$$W_2(2) + W_2(3)$$

for the provided μ_j, $j=1,2,3$, and a "ring" topology for the nodes.

This last topology implies the following routing matrix P:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and therefore the quantities π_j introduced in the discussion on CQNs are computed as follows:

$$\left\{ \begin{array}{l} (\pi_1 \; \pi_2 \; \pi_3) = (\pi_1 \; \pi_2 \; \pi_3) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \\ \sum_{j=1}^{3} \pi_j = 1 \end{array} \right.$$

$$\Rightarrow \quad \pi_1 = \pi_2 = \pi_3 = 1/3$$

Then, applying the recursion:

$$\left\{ \begin{array}{ll} W_m(j) = \dfrac{1}{\mu_j} + \dfrac{(m-1)\pi_j W_{m-1}(j)}{\mu_j \sum_{i=1}^{3} \pi_i W_{m-1}(i)}, & \forall j \\ W_1(j) = 1/\mu_j, & \forall j \end{array} \right.$$

that we derived during the discussion of MVA, we obtain:

$$W_2(2) = \frac{1}{\mu_2} + \frac{(2-1)\,\frac{1}{3}\,W_1(2)}{\mu_2\left(\frac{1}{3}W_1(1) + \frac{1}{3}W_1(2) + \frac{1}{3}W_1(3)\right)} =$$

$$= \frac{1}{\mu_2}\left[1 + \frac{1/\mu_2}{1/\mu_1 + 1/\mu_2 + 1/\mu_3}\right] =$$

$$= \frac{1}{1}\left[1 + \frac{1/1}{1/1.5 + 1/1 + 1/1.2}\right] = 1.4 \text{ min}$$

$$W_2(3) = \frac{1}{\mu_3} + \frac{(2-1)\,\frac{1}{3}\,W_1(3)}{\mu_3\left[\frac{1}{3}W_1(1) + \frac{1}{3}W_1(2) + \frac{1}{3}W_1(3)\right]} =$$

$$= \frac{1}{\mu_3}\left[1 + \frac{1/\mu_3}{1/\mu_1 + 1/\mu_2 + 1/\mu_3}\right] = \frac{1}{1.2}\left[1 + \frac{1/1.2}{1/1.5 + 1/1 + 1/1.2}\right] =$$

$$\simeq 1.11 \text{ min}$$

and the sought expected time is

$$W_2(2) + W_2(3) \simeq 1.4 + 1.11 \simeq 2.51 \text{ min}.$$

A less mechanistic way to derive the above results in the considered case, is as follows:

Let also:

- $L_m(j)$ denote the expected number of customers at station j for an m-customer network
- $TH(m)$ denote the throughput for an m-customer network
- $(T(m))$ denote the total expected time for going through all three nodes

Then, from Little's law (applied to the entire network):

$$TH(1) = \frac{1}{CT(1)} = \frac{1}{1/\mu_1 + 1/\mu_2 + 1/\mu_3}$$

Also, application of Little's law at nodes 2 and 3 gives:

$$L_1(2) = TH(1) \cdot 1/\mu_2 \quad ; \quad L_1(3) = TH(1) \cdot 1/\mu_3$$

and from the arrival theorem:

$$W_2(2) = \frac{1}{\mu_2}\left(1 + L_1(2)\right)$$

$$W_2(3) = \frac{1}{\mu_3}\left(1 + L_1(3)\right)$$

which give

$$W_2(2) = \frac{1}{\mu_2}\left[1 + \frac{1/\mu_2}{1/\mu_1 + 1/\mu_2 + 1/\mu_3}\right]$$

$$W_2(3) = 1/\mu_3\left[1 + \frac{1/\mu_3}{1/\mu_1 + 1/\mu_2 + 1/\mu_3}\right]$$

as before.

For part (b), just compute also

$$W_2(1) = \frac{1}{\mu_1}\left[1 + \frac{1/\mu_1}{1/\mu_1 + 1/\mu_2 + 1/\mu_3}\right] =$$

$$= \frac{1}{1.5}\left[1 + \frac{1/1.5}{1/1.5 + 1/1 + 1/2}\right] \simeq 0.84 \text{ min}$$

Then

$$CT(2) = W_2(1) + W_2(2) + W_2(3) \simeq 0.84 + 2.51 =$$

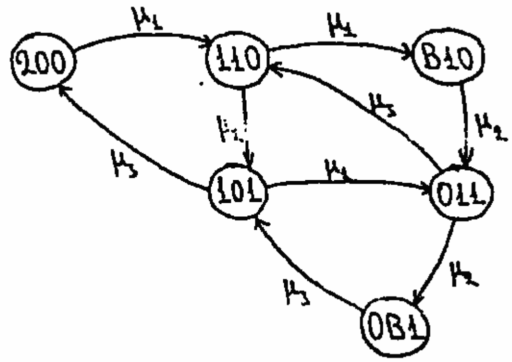
$$= 3.35 \text{ min}$$

and

$$TH(2) = \frac{2}{CT(2)} = \frac{2}{3.35} = 0.597 \text{ min}^{-1} \simeq 0.6 \text{ min}^{-1}$$

Problem 8.15:

a)



b) Let's first calculate the stationary state probabilities:

$$\left.\begin{aligned}
\mu_1\pi(200) &= \mu_3\pi(101)\\
(\mu_1+\mu_2)\pi(110) &= \mu_1\pi(200)+\mu_3\pi(011)\\
\mu_2\pi(B10) &= \mu_1\pi(110)\\
(\mu_2+\mu_3)\pi(011) &= \mu_2\pi(B10)+\mu_1\pi(101)\\
\mu_3\pi(0B1) &= \mu_2\pi(011)\\
(\mu_1+\mu_3)\pi(101) &= \mu_2\pi(110)+\mu_3\pi(0B1)
\end{aligned}\right\} \Rightarrow
\begin{aligned}
1.5\pi(200) &= 1.2\pi(101)\\
2.5\pi(110) &= 1.5\pi(200)+1.2\pi(011)\\
\pi(B10) &= 1.5\pi(110)\\
2.2\pi(011) &= \pi(B10)+1.5\pi(101)\\
1.2\pi(0B1) &= \pi(011)\\
2.7\pi(101) &= \pi(110)+1.2\pi(0B1)
\end{aligned}$$

We also have that

$$\pi(200)+\pi(110)+\pi(B10)+\pi(101)+\pi(011)+\pi(0B1)=1$$

Solving the above system of linear equations we get:

$$\pi(200)=\frac{592}{5587}$$

$$\pi(101)=\frac{740}{5587}$$

$$\pi(110)=\frac{888}{5587}$$

$$\pi(011)=\frac{1110}{5587}$$

$$\pi(0B1)=\frac{925}{5587}$$

$$\pi(B10)=\frac{1332}{5587}$$

P[any one node in the system is blocked] $= \pi(0B1) + \pi(B10) =$

$$= \frac{925}{5587} + \frac{1332}{5587} = \frac{2257}{5587} \simeq .404$$

c) P[only one node in the system is actually processing a customer] =

$$= \pi(200) + \pi(B10) + \pi(0B1) = \frac{592}{5587} + \frac{2257}{5587} = \frac{2849}{5587} \simeq 0.51$$

It is also interesting to compute the throughput attained by this new configuration and compare it to the ~~configuration~~ throughput attained by the configuration of Problem 8.14.

This new throughput is given by:

$$TH' = \mu_2 \left[\pi(101) + \pi(011) + \pi(0B1) \right] =$$

$$= 1.2 \left[\frac{740}{5587} + \frac{1110}{5587} + \frac{925}{5587} \right] \simeq$$

$$\simeq 0.596 \ \text{min}^{-1}$$

In this case, the blocking taking place in this new configuration does not have a substantial impact on the system throughput. But, in general, the reduction of throughput due to introduced blocking effects can be significant.

* For an explanation of the throughput equality observed between problems 8.14 and 8.15 see the very pertinent(!) remarks provided in the next 3 pages; these pages are the solution of problem 8.15 by ~~[illegible]~~ an ex-student of this course.

$$L_1(1) = \lambda_1 \cdot W_1(1) = \lambda_1 \cdot \frac{1}{1.5}$$

$$L_2(1) = \lambda_1 \cdot W_2(1) = \lambda_1$$

$$L_3(1) = \lambda_1 \cdot W_3(1) = \lambda_1 \cdot \frac{1}{1.2}$$

$$L_1(1) + L_2(1) + L_3(1) = \lambda_1 \left(\frac{1}{1.5} + 1 + \frac{1}{1.2}\right) = 1$$

→ $\lambda_1 = 0.4$

→ $L_1(1) = 0.2667$. $L_2(1) = 0.4$. $L_3(1) = 0.3333$

For N=2.

$$W_1(2) = \frac{1}{\mu_1} + \frac{1}{\mu_1} \cdot L_1(1) = 0.8445$$

$$W_2(2) = \frac{1}{\mu_2} + \frac{1}{\mu_2} \cdot L_2(1) = 1.4$$

$$W_3(2) = \frac{1}{\mu_3} + \frac{1}{\mu_3} \cdot L_3(1) = 1.1111$$

$$L_1(2) = \lambda_2 \cdot W_1(2) = \lambda_2 \cdot 0.8445$$

$$L_2(2) = \lambda_2 \cdot W_2(2) = \lambda_1 \cdot 1.4$$

$$L_3(2) = \lambda_2 \cdot W_3(2) = \lambda_2 \cdot 1.1111$$

→ $L_1(2) + L_2(2) + L_3(2) = \lambda_2(0.8445 + 1.4 + 1.1111) = 2$

$\lambda_2(0.8445 + 1.4 + 1.1111) = 2$

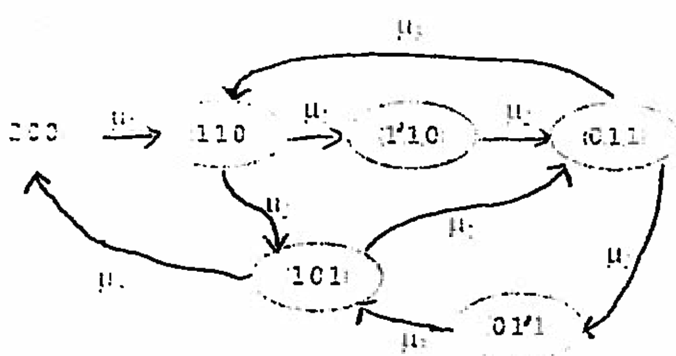
→ $\lambda_2 = 0.596$

(a) $W_2(2) + W_3(2) = 1.4 + 1.1111 = 2.5111$

(b) $\lambda_2 = 0.596$

8.15 in text book

a)



In state space, number with prime indicates job blocked by subsequent node.

b) Following flow balance equations should be satisfied in the equilibrium.

for state (2,0,0) $\mu_1 \cdot \pi(2,0,0) = \mu_3 \cdot \pi(1,0,1)$

for state (1,1,0) $\mu_1 \cdot \pi(1,1,0) + \mu_2 \cdot \pi(1,1,0) = \mu_1 \cdot \pi(2,0,0) + \mu_3 \cdot \pi(0,1,1)$

for state (1',1,0) $\mu_2 \cdot \pi(1',1,0) = \mu_1 \cdot \pi(1,1,0)$

for state (0,1,1) $(\mu_2 + \mu_3) \cdot \pi(0,1,1) = \mu_2 \cdot \pi(1',1,0) + \mu_1 \cdot \pi(1,0,1)$

for state (1,0,1) $(\mu_1 + \mu_3) \cdot \pi(1,0,1) = \mu_2 \cdot \pi(1,1,0) + \mu_3 \cdot \pi(0,1',1)$

for state (0,1',1) $\mu_3 \cdot \pi(0,1',1) = \mu_2 \cdot \pi(0,1,1)$

by distribution property $\sum_{s \in S} \pi(s) = 1$.

if we solve the above equations, we can get stationary probability for each state.

I got the following result by several matrix operation using Matlab.

$$\begin{bmatrix} \pi(2,0,0) \\ \pi(1,1,0) \\ \pi(1',1,0) \\ \pi(0,1,1) \\ \pi(1,0,1) \\ \pi(0,1',1) \end{bmatrix} = \begin{bmatrix} 0.1060 \\ 0.1589 \\ 0.2383 \\ 0.1987 \\ 0.1325 \\ 0.1656 \end{bmatrix}$$

→ <u>probability that any customers are blocked:</u>

$$\underline{\pi(1',1,0) + \pi(0,1',1) = 0.2383 + 0.1656 = 0.4039}$$

Further, I think alternative way using special structure of this problem.

What if we replace (1',1,0) (respectively (0,1',1)) with (0,2,0) (respectively (0,0,2))? Since there is only two customers in the system, the state diagram of (a) will not change at all. In other words, we can substitute (0,2,0) and (0,0,2) for (1',1,0) and (0,1',1) respectively without any change of state diagram. Moreover, the changed state diagram is nothing more than state diagram of network without any capacity constraints. Therefore, we can analyze this system as the normal closed network, while considering state (0,2,0) and (0,0,2) as blocking situation (1'1,0) and (0,1',1) respectively.

If this is normal closed network, we can assume that stationary distribution is product form as follows. Since relative arrival rates are same, we can assume that λ=1.

→ $\rho_1 = \dfrac{1}{1.5} = 0.667$, $\rho_2 = 1$, $\rho_3 = 0.833$

→ $\pi(n_1, n_2, n_3) = A \cdot 0.667^{n_1} \cdot 0.833^{n_3}$

Let's get a normalization constant A.

$$\sum_{(n_1,n_2,n_3)} \pi(n_1,n_2,n_3) = A\cdot \sum_{(n_1,n_2,n_3)} 0.667^{n_2}\cdot 0.833^{n_3} = 1$$

$$\rightarrow \begin{aligned} &A(0.667^2\cdot 0.833^0 + 0.667^0\cdot 0.833^0 + 0.667^0\cdot 0.833^2 \\ &+ 0.667^1\cdot 0.833^0 + 0.667^1\cdot 0.833^1 + 0.667^0\cdot 0.833^1) = 1 \end{aligned}$$

$\rightarrow A = 0.2384$

In this new state diagram, (0,2,0) and (0,0,2) indicate blocking situation.

$$\pi(0,2,0) + \pi(0,0,2) = 0.2384\cdot(1+0.833^2) = 0.4038$$

(e)

States that only one node actually does processing are:

$\pi(2,0,0) + \pi(1',1,0) + \pi(0,1',1) = 0.1060 + 0.2383 + 0.1656 = 0.5099$ from 1st approach. On the other hand, from 2nd approach,

$$\pi(2,0,0) + \pi(0,2,0) + \pi(0,0,2) = 0.2384(0.667^2 + 1 + 0.833^2) = 0.5097$$

We can see both approaches produce same results.

Problem B

$$\Pr(W=t) = \sum_{n=0}^{\infty} \Pr(W=t \mid X=n)\cdot \Pr(X=n)$$

, where X is the number of customers in the system when a new customer is arriving. Since service time is exponential, when a new customer joins the queue, the current service can be renewed like starting over because of memoryless property.

$$\rightarrow \Pr(W=t \mid X=n) = Erlang(n+1,\mu) = \frac{\mu^{n+1}\cdot t^n\cdot e^{-\mu t}}{n!}$$

Note that in order to get through the entire system, n+1 processes should be repeated, including the new coming customer itself.

$$\Pr(W=t) = \sum_{n=0}^{\infty} \frac{\mu^{n+1}\cdot t^n\cdot e^{-\mu t}}{n!}\cdot a_n = \sum_{n=0}^{\infty} \frac{\mu^{n+1}\cdot t^n\cdot e^{-\mu t}}{n!}\cdot \beta^n(1-\beta)$$

$$= \mu(1-\beta)\cdot e^{-\mu t}\sum_{n=0}^{\infty}\frac{(\mu\cdot t\cdot \beta)^n}{n!} = \mu(1-\beta)\cdot e^{-\mu t}\cdot e^{\mu t\beta}$$

(b)

3.28) Since customers are serviced on a FCFS basis and arrivals are Poisson, we can treat the proc. time of these customers as a mixture of the two exp. service times with mixing probabilities $\lambda_1/\lambda_1+\lambda_2$ and $\lambda_2/\lambda_1+\lambda_2$. Also, the entire queue becomes an $M/G/1$ queue with total arrival rate $\lambda = \lambda_1+\lambda_2$ and traffic intensity $\rho = (\lambda_1+\lambda_2)E[T]$ where T denotes the mixing r.v described above. From the MVA of the $M/G/1$ queue:

$$W_q = \frac{1+SCV[T]}{2}\,\frac{u(T)}{1-u(T)}\,E[T].$$

$$SCV[T] = \frac{Var[T]}{E^2[T]} = \frac{E[T^2]-E^2[T]}{E^2[T]} = \frac{E[T^2]}{E^2[T]} - 1$$

$$u[T] = \rho = (\lambda_1+\lambda_2)E[T]$$

$$\Rightarrow W_q = \frac{E[T^2]}{2E^2[T]}\,\frac{(\lambda_1+\lambda_2)E[T]}{1-(\lambda_1+\lambda_2)E[T]}\,E[T] = \frac{E[T^2]}{2}\,\frac{\lambda_1+\lambda_2}{1-\rho} \quad (1)$$

$$E[T^2] = \frac{\lambda_1}{\lambda_1+\lambda_2}E[T_1^2] + \frac{\lambda_2}{\lambda_1+\lambda_2}E[T_2^2] =$$

$$\frac{\lambda_1}{\lambda_1+\lambda_2}\left(Var[T_1]+E^2[T_1]\right) + \frac{\lambda_2}{\lambda_1+\lambda_2}\left(Var[T_2]+E^2[T_2]\right) =$$

$$= \frac{\lambda_1 E^2[T_1]}{\lambda_1+\lambda_2}\left(scv[T_1]+1\right) + \frac{\lambda_2 E^2[T_2]}{\lambda_1+\lambda_2}\left(scv[T_2]+1\right) =$$

nce T_1 & T_2 are exp

$$= \frac{2\rho_1 E[T_1]}{\lambda_1+\lambda_2} + \frac{2\rho_2 E[T_2]}{\lambda_1+\lambda_2} \quad (2)$$

From (1) and (2),

$$W_q = \frac{\rho_1 E[T_1] + \rho_2 E[T_2]}{1-\rho} = \frac{\rho_1/\mu_1 + \rho_2/\mu_2}{1-\rho}$$

Then, 3.38 follows immediately from Little's law.

3.33) According to Eq. 3.43 in the text,

$$W_q^{(i)} = \frac{\sum_{k=1}^{r} \rho_k/\mu_k}{\left(1-\sum_{k=1}^{i-1}\rho_k\right)\left(1-\sum_{k=1}^{i}\rho_k\right)} \qquad \forall i=1,\dots,r$$

where $\rho_k = \lambda_k/\mu_k$.

For $\mu_k = \mu$, $\forall k$, the above equation implies

$$W_q^{(i)} = \frac{\sum_{k=1}^{r}\lambda_k}{\left(\mu-\sum_{k=1}^{i-1}\lambda_k\right)\left(\mu-\sum_{k=1}^{i}\lambda_k\right)} = \frac{1}{\left(1-\sum_{k=1}^{i-1}\rho_k\right)\left(1-\sum_{k=1}^{i}\rho_k\right)}\,\frac{\rho}{\mu}$$

where we have set $\rho = \sum_{k=1}^{r}\rho_k = \frac{1}{\mu}\sum_{k=1}^{r}\lambda_k$

To prove the requested result, we need to show that

$$\sum_{i=1}^{r}\frac{\lambda_i}{\lambda}\,\frac{1}{\left(1-\sum_{k=1}^{i-1}\rho_k\right)\left(1-\sum_{k=1}^{i}\rho_k\right)} = \frac{1}{1-\rho} \qquad \text{where } \lambda = \sum_{k=1}^{r}\lambda_k$$

We shall prove this result by induction. Obviously the result holds for $r=1$. Next assume that the result holds up to $r-1$, and we shall establish it for r. Also, to simplify notation, let us set

$$\frac{1}{\left(1-\sum_{k=1}^{i-1}\rho_k\right)\left(1-\sum_{k=1}^{i}\rho_k\right)} \equiv A_i.$$

Then we have:

$$\sum_{i=1}^{r}\frac{\lambda_i}{\lambda}A_i = \sum_{i=1}^{r-1}\frac{\lambda_i}{\lambda}A_i + \frac{\lambda_r}{\lambda}A_r = \frac{\hat\lambda}{\lambda}\sum_{i=1}^{r-1}\frac{\lambda_i}{\hat\lambda}A_i + \frac{\lambda_r}{\lambda}A_r \qquad (1)$$

where $\hat\lambda_r = \sum_{k=1}^{r-1}\lambda_k$. Applying the induction hypothesis to (1), we get:

$$\sum_{i=1}^{r}\frac{\lambda_i}{\lambda}A_i = \frac{\hat\lambda}{\lambda}\,\frac{1}{1-\hat\rho} + \frac{\lambda_r}{\lambda}\,\frac{1}{(1-\hat\rho)(1-\rho)} \qquad \left(\text{where } \hat\rho = \sum_{k=1}^{r-1}\rho_k\right)$$

$$= \frac{\hat\lambda(1-\rho)+\lambda_r}{\lambda(1-\hat\rho)(1-\rho)}$$

But $\hat\lambda(1-\rho)+\lambda_r = \hat\lambda(1-\hat\rho-\rho_r)+\lambda_r = \hat\lambda(1-\hat\rho) - \hat\lambda\frac{\lambda_r}{\mu} + \lambda_r = \hat\lambda(1-\hat\rho)+\lambda_r(1-\hat\rho) = \lambda(1-\hat\rho),$

3.35) This problem can be solved easily by applying the result of Problem 3.33. Then, we have:

$$\rho = \lambda/\mu = \lambda\tau = 10 \times \frac{5.5}{60} = \frac{5.5}{6}$$

$$W_q = \frac{\rho}{1-\rho}\tau = \frac{5.5/6}{1-5.5/6} 5.5 = \frac{5.5}{0.5} 5.5 = 60.5 \text{ min}$$

$$W = W_q + \tau = 60.5 + 5.5 = 66 \text{ min}$$

3.36) In this case, we can obtain first the average number of customers from each class, $L^{(i)}$, $i = 1, 2, 3$, through the formula provided by Section 3.4.3 of your textbook. Then

$$W = \sum_{i=1}^{3} \frac{\lambda_i}{\lambda} W^{(i)} = \sum_{i=1}^{3} \frac{\lambda_i}{\lambda} \frac{L^{(i)}}{\lambda_i} = \frac{1}{\lambda} \sum_{i=1}^{3} L^{(i)}$$

where the second equation is obtained by applying Little's law at each class.

We have
$$L^{(i)} = \frac{\rho_i}{1-\sigma_{i-1}} + \frac{\lambda_i \sum_{j=1}^{i} \lambda_j E[S_j^2]}{2(1-\sigma_{i-1})(1-\sigma_i)}$$

Hence, the necessary computations can be organized as follows:

First: $E[S_j^2] = E(S^2) = Var[S] + E^2[S] = 2E^2[S] =$
$$= 2 \times \left(\frac{5.5}{60}\right)^2 = 0.0168$$

In the above computation we have taken into consideration that service times are exponentially distributed and therefore $SCV[S] = 1.0$

Also:

$\lambda_1 = 10/5 = 2 \text{ hr}^{-1}$; $\lambda_2 = 10\frac{3}{10} = 3 \text{ hr}^{-1}$; $\lambda_3 = 10 \times \frac{1}{2} = 5 \text{ hr}^{-1}$

$\rho_1 = \lambda_1\tau = 5.5/30$; $\rho_2 = \lambda_2\tau = 5.5/20$; $\rho_3 = \lambda_3\tau = \frac{5.5}{12}$

$\sigma_1 = \rho_1 = 5.5/30$; $\sigma_2 = \rho_1 + \rho_2 = 5.5/12$; $\sigma_3 = \rho_1 + \rho_2 + \rho_3 = \frac{5.5}{6}$

$$L^{(1)} = \frac{\rho_1}{1-\sigma_0} + \frac{\lambda_1^2 E[S]}{2(1-\sigma_0)(1-\sigma_1)} = \frac{5.5}{30} + \frac{2^2 \times 0.0168}{2(1-5.5/30)} \simeq 0.2245$$

$$L^{(2)} = \frac{\rho_2}{1-\sigma_1} + \frac{\lambda_2(\lambda_1+\lambda_2)E[S]}{2(1-\sigma_1)(1-\sigma_2)} = \frac{5.5/20}{1-5.5/30} + \frac{3(2+3)\,0.0168}{2(1-5.5/30)(1-5.5/12)} \simeq 0.6216$$

$$L^{(3)} = \frac{\rho_3}{1-\sigma_2} + \frac{\lambda_3 \cdot \lambda \cdot E[S]}{2(1-\sigma_2)(1-\sigma_3)} = \frac{5.5/12}{1-5.5/12} + \frac{5\times 10\times 0.0168}{2(1-5.5/12)(1-5.5/6)} \simeq 10.1508$$

Finally, $W = \frac{1}{10}(0.2245 + 0.6216 + 10.1508) \simeq 1.1$ hrs $= 65.98$ min

($\simeq 66$ min, which was the result in Problem 3.35)

For an explanation of the results in Problems 3.33, 3.35 and 3.36, consider the execution of the corresponding queues under any given sample path. This sample path can be represented by (i) the sequences of the interarrival times for the different job classes, and (ii) a single sequence of proc. times drawn from the common distribution of all classes. Then, a simulation of the queues according to this sample path for the case of problem 3.33 would reveal that irrespective of how the server chooses among the waiting classes, the variable TW that tracks the total waiting time across all classes will evolve in the same manner.

In the case of comparing the M/M/1 queue with the preemptive-priorities queue, the above argument is not as clear because of effects arising due to preemption. But these effects can be circumvented by acknowledging the memoryless property of exp distribution and modifying thus the simulation process that is suggested above so that the simulator assigns a new proc. time for a part in service upon a new arrival.

There is also a conceptual affinity of these problems with the analysis that establishes the optimality of the Shortest Expected Proc Time (SEPT) dispatching rule for minimizing the expected average cycle time in case of jobs with exponential proc. times.

Problem C

From the MVA of the M/G/1 queue performed in class, we know that

$$L = \lambda W = \lambda (W_q + E[S]) = \lambda W_q + \lambda E[S] =$$

$$= \lambda \frac{1+C_B^2}{2} \frac{\rho}{1-\rho} E[S] + \rho = \frac{1+C_B^2}{2} \frac{\rho^2}{1-\rho} + \rho$$

Also, as discussed in class

$$L = \Pi'(z)\big|_{z=1}$$

Hence, our task is to show that

$$\Pi'(z)\big|_{z=1} = \frac{1+C_B^2}{2} \frac{\rho^2}{1-\rho} + \rho \qquad (1)$$

We also know that

$$\Pi(z) = \frac{(1-\rho)(1-z)K(z)}{K(z)-z} = \frac{K(z) - zK(z)}{K(z)-z}(1-\rho) \qquad (2)$$

where $K(z)$ is the probability generating function for the distribution characterizing the number of arrivals within a service time period.

From (2),

$$\Pi'(z) = (1-\rho) \frac{(K(z)-z)\left[K'(z) - K(z) - zK'(z)\right] - (K(z) - zK(z))(K'(z)-1)}{(K(z)-z)^2}$$

$$= (1-\rho) \frac{K(z) - K^2(z) - (z - z^2)K'(z)}{(K(z)-z)^2} \qquad (3)$$

From (3), taking into consideration that $K(1)=1$, we see that $\Pi'(z)|_{z=1}$ takes the undefined form $0/0$. So, we proceed by applying L'Hopital's rule (twice) on the quantity:

$$A = \frac{K(z) - K^2(z) - (z - z^2) K'(z)}{(K(z) - z)^2}$$

From the first application of this rule we get:

$$\lim_{z\to 1} A = \lim_{z\to 1} \frac{K'(z) - 2K(z)K'(z) - (1-2z)K'(z) - (z-z^2)K''(z)}{2(K(z)-z)(K'(z)-1)}$$

$$= \lim_{z\to 1} \frac{[2z - 2K(z)]K'(z) - (z-z^2)K''(z)}{2(K(z)-z)(K'(z)-1)}$$

From the second application we have

$$\lim_{z\to 1} A = \lim_{z\to 1} \frac{2[1-K'(z)]K'(z) + 2[z-K(z)]K''(z) - (1-2z)K''(z) - (z-z^2)K'''(z)}{2[(K'(z)-1)(K'(z)-1) + (K(z)-z)K''(z)]} =$$

$$= \frac{2[1-\rho]\rho + K''(1)}{2(\rho-1)^2} = \frac{K''(1)}{2(1-\rho)^2} + \frac{\rho}{1-\rho} \quad (4)$$

So, from (3) and (4) we get

$$\Pi'(z)|_{z=1} = \frac{K''(1)}{2(1-\rho)} + \rho \quad (5)$$

In the above, we have used the fact that $K'(z)|_{z=1} = \rho$, that was established in class.

To proceed from (5), we need to express $K''(1)$ in terms of the problem parameters. For this, we proceed as follows:

We know that

$$K(z) = \sum_{i=0}^{\infty} k_i z^i \Rightarrow \frac{dK(z)}{dz} = \sum_{i=1}^{\infty} i k_i z^i \Rightarrow$$

$$\Rightarrow \frac{d^2K(z)}{dz^2} = \sum_{i=2}^{\infty} i(i-1) k_i z^{i-2} \Rightarrow$$

$$\Rightarrow \left.\frac{d^2K(z)}{dz^2}\right|_{z=1} = \sum_{i=2}^{\infty} i^2 k_i - \sum_{i=2}^{\infty} i k_i =$$

$$= E[A^2] - k_1 - E[A] + k_1 = E[A^2] - E[A] =$$

$$= Var[A] + E[A]^2 - E[A] \quad (6)$$

where A denotes the random arrivals over a service time period.

In class, we argued that $E[A] = \lambda E[S] = \rho$.

A formal proof for this result is by taking conditional expectations:

$$E[A] = E[E[A|S]] = E[\lambda S] = \lambda E[S] = \rho.$$

To compute the quantity $Var[A]$ that appears in (6), we use the result of Eq (5.6) in the provided notes on the $M/G/1$ queue (also see next page):

$$Var[A] = \rho + \lambda^2 \sigma_B^2 \quad (7)$$

Hence, $K''(1) = \rho + \lambda^2\sigma_B^2 + \rho^2 - \rho = \rho^2 + \lambda^2\sigma_B^2 =$

$$= \rho^2 + (\lambda E)^2 \frac{\sigma_B^2}{E^2} = \rho^2 + \rho^2 C_B^2 = \rho^2(1 + C_B^2) \quad (8)$$

But then, (8) and (5) imply that

$$\Pi'(z)|_{z=1} = \frac{1+C_B^2}{2}\frac{\rho^2}{1-\rho} + \rho$$ which proves (1).

Proving Eq 5.6

$$\mathrm{Var}[A] = E[A^2] - E^2[A] =$$

$$= E\left[E[A^2|S]\right] - E^2\left[E[A|S]\right] =$$

$$= E\left[\mathrm{Var}[A|S] + E^2[A|S]\right] - E^2\left[E[A|S]\right] =$$

$$= E\left[\mathrm{Var}[A|S]\right] + E\left[E^2[A|S]\right] - E^2\left[E[A|S]\right] =$$

$$= E\left[\mathrm{Var}[A|S]\right] + \mathrm{Var}\left[E[A|S]\right] =$$

$$= E[\lambda S] + \mathrm{Var}[\lambda S] =$$

$$= \lambda E[S] + \lambda^2 \mathrm{Var}[S] =$$

$$= \rho + \lambda^2 \sigma_B^2$$

Problem d

To answer the posed question, we need first to compute the mean effective processing time. For this, we have

$$E[T_{eff}] = E[T_{part} + T_{rework}] = E[T_{part}] + E[T_{rework}]$$

$$E[T_{part}] = 2 \text{ min}$$

$$E[T_{rework}] = E[T_{rework} \mid \text{only part 1 defective}] \cdot p_1 * (1-p_2) +$$
$$+ E[T_{rework} \mid \text{" " 2 "}] \cdot (1-p_1) * p_2 +$$
$$+ E[T_{rework} \mid \text{both parts defective}] \cdot p_1 * p_2 +$$
$$+ E[T_{rework} \mid \text{no part defective}] \cdot (1-p_1) \cdot (1-p_2)$$

$$E[T_{rework} \mid \text{no part defective}] = 0 \text{ min}$$

$$E[T_{rework} \mid \text{only part 1 defective}] = \frac{1}{r_1} = \frac{1}{0.2 \text{ min}^{-1}} = 5 \text{ min}$$

$$E[T_{rework} \mid \text{only part 2 defective}] = \frac{1}{r_2} = \frac{1}{0.1 \text{ min}^{-1}} = 10 \text{ min}$$

$$E[T_{rework} \mid \text{both parts defective}] =$$
$$= E[\max(T^1_{rework}, T^2_{rework}) \mid \text{both parts defective} \wedge T^1_{rework} < T^2_{rework}] \cdot$$
$$\cdot p(T^1_{rework} < T^2_{rework} \mid \text{both parts defective}) +$$
$$E[\max(T^1_{rework}, T^2_{rework}) \mid \text{both parts defective} \wedge T^1_{rework} > T^2_{rework}] \cdot$$
$$p(T^1_{rework} > T^2_{rework} \mid \text{both parts defective}) =$$
$$= E[T^1_{rework} + (T^2_{rework} - T^1_{rework}) \mid \text{both parts defective}] \cdot \frac{r_1}{r_1+r_2} +$$
$$+ E[T^2_{rework} + (T^1_{rework} - T^2_{rework}) \mid \text{both parts defective}] \cdot \frac{r_2}{r_1+r_2} =$$
$$= \left(\frac{1}{r_1+r_2} + \frac{1}{r_2}\right)\frac{r_1}{r_1+r_2} + \left(\frac{1}{r_1+r_2} + \frac{1}{r_1}\right)\frac{r_2}{r_1+r_2} =$$
$$\frac{1}{r_1+r_2}\left(1 + \frac{r_1}{r_2} + \frac{r_2}{r_1}\right) = \frac{1}{0.2 \text{ min}^{-1} + 0.1 \text{ min}^{-1}}\left(1 + \frac{0.2}{0.1} + \frac{0.1}{0.2}\right) = 11.67 \text{ min}$$

Hence, $E[T_{rework}] = 5 \cdot 0.3 \cdot 0.8 +$

$10 \cdot 0.7 \cdot 0.2 +$

$11.67 \cdot 0.3 \cdot 0.2 \simeq 3.3 \text{ min}$

and

$E[T_{eff}] = 2 + 3.3 = 5.3 \text{ min}$

Therefore, the effective processing capacity is: $60/5.3 = 11.32$ units/hr.

(e)

Problem 4 (20 points): A service station is processing the commingled stream of two part types, each of which arrives according to a Poisson process with rate λ_i. Parts are processed on a FCFS basis, and the expected service time for either part is equal to t_p time units, but when switching from one part type to the other, there is an additional deterministic set-up time equal to t_s time units. Provide the stability condition for this station; your response must expressed in terms of the data set provided above.

The probability for a setup between two consecutive jobs can be written as

$$P(\text{setup}) = P(\text{previous job is type 1}) \cdot P(\text{setup} \mid \text{prev. job type 1}) + P(\text{previous job is type 2}) \cdot P(\text{setup} \mid \text{prev. job type 2})$$

Taking into consideration the Poisson and independent nature of the two arrival processes, we get:

$$P(\text{setup}) = \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{\lambda_1}{\lambda_1+\lambda_2} = \frac{2\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}$$

Hence, the expected proc time for any part, when accounting for the potential setups, is:

$$t_e = t_p + P(\text{setup})\, t_s = t_p + \frac{2\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2} t_s$$

For stability, we need:

$$(\lambda_1+\lambda_2)\, t_e < 1 \implies (\lambda_1+\lambda_2)\, t_p + \frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2} t_s < 1$$

(8)

Problem 2 (20 points): A machine can experience two types of failure. Both types of failure can occur only when the machine is operational (i.e.. failures are "operation-driven" and not "time-driven"). they occur independently from each other. and their occurrences follow Poisson processes with corresponding rates λ_i, $i = 1, 2$. Also. the corresponding MTTRs (mean time to repare) are equal to t_i, $i = 1, 2$. Answer the following questions:

i. What is the *availablity* of this machine?

ii. If both types of failure are non-destructive and the "nominal" processing times (i.e.. the times that are required for the processing of the parts without accounting for the downtimes due to failures) for this machine are uniformly distributed over the interval $[a, b]$, what is the expected number of failures that take place during the processing of a single part?

In your response. consider that all the referred quantities are given in consistent units.

(i) We know that $A = \dfrac{MTTF}{MTTF + MTTR}$

where
- MTTF = mean time to failure
- MTTR = " " " repair

Since each failure type occurs according to a Poisson distribution (while the machine is operational) and these two processes are mutually independent,

$$MTTF = \frac{1}{\lambda_1 + \lambda_2}$$

On the other hand

$$MTTR = \frac{\lambda_1}{\lambda_1 + \lambda_2} t_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} t_2 = \frac{\lambda_1 t_1 + \lambda_2 t_2}{\lambda_1 + \lambda_2}$$

finally

$$A = \frac{1/(\lambda_1 + \lambda_2)}{1/(\lambda_1 + \lambda_2) + \frac{\lambda_1 t_1 + \lambda_2 t_2}{\lambda_1 + \lambda_2}} = \frac{1}{1 + \lambda_1 t_1 + \lambda_2 t_2}$$

(ii) Let r.v. T = the part proc time
and r.v. N = # of failures that take place during the part processing

Then,

$$E[N] = E\left[E[N \mid T]\right] = E[(\lambda_1+\lambda_2)T] =$$
$$= (\lambda_1+\lambda_2)\,E[T] = (\lambda_1+\lambda_2)\,\frac{a+b}{2}$$

(iii) The <u>effective</u> proc time for any job going through this workstation can be represented by a r.v.

$$T_{eff} = T_p + \sum_{i=1}^{N} T_{f_i}$$

where r.v. T_p denotes the actual proc. times, r.v N denotes the random number of failures experienced during processing and T_{f_i} is the random duration of the i-th failure. T_{f_i} is modelled as a "mixture" of the two r.v's modelling the downtimes w.r.t. each of the two failure types with respective mixing probabilities $P_1 = \frac{\lambda_1}{\lambda_1+\lambda_2}$; $P_2 = \frac{\lambda_2}{\lambda_1+\lambda_2}$. Hence,

$$t_e = E[T_{eff}] = E[T_p] + E[N]\,E[T_f] =$$
$$= E[T_p] + (\lambda_1+\lambda_2)\,E[T_p]\left[\frac{\lambda_1}{\lambda_1+\lambda_2}t_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}t_2\right]$$
$$= E[T_p]\left[1 + \lambda_1 t_1 + \lambda_2 t_2\right] \quad \left(= E[T_p]/A\right)$$

Hence, for stability: $r_a t_{eff} < 1 \Rightarrow r_a < 1/t_{eff} = A/E[T_p] = A r_p$
where $r_p = 1/E[T_p]$