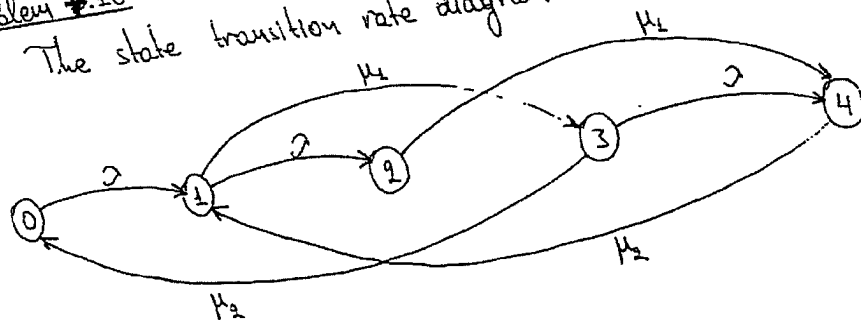


SOLUTIONS FOR HW #3

Problem 7.16

a) The state transition rate diagram is



b) For $\lambda=1$, $\mu_1=\frac{3}{2}$, $\mu_2=\frac{1}{4}$,

$$Q = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -5/2 & 1 & 3/2 & 0 \\ 0 & 0 & -3/2 & 0 & 3/2 \\ 1/4 & 0 & 0 & -11/4 & 1 \\ 0 & 1/4 & 0 & 0 & -1/4 \end{bmatrix}$$

The stationary state probabilities are calculated as follows:

$$\pi Q = [0 \ 0 \ 0 \ 0 \ 0]$$

and

$$\pi \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1$$

That is

$$[\pi_0 \ \pi_1 \ \pi_2 \ \pi_3 \ \pi_4] \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & -5/2 & 1 & 3/2 & 0 \\ 1 & 0 & -3/2 & 0 & 3/2 \\ 1 & 0 & 0 & -11/4 & 1 \\ 1 & 1/4 & 0 & 0 & -1/4 \end{bmatrix} = [1 \ 0 \ 0 \ 0 \ 0]$$

The result is $\pi = [0.2357 \ 0.2469 \ 0.1646 \ 0.1347 \ 0.2181]$
(this is solved by computer)

c) Let the state at steady state be denoted by X . We want $E[X|X \neq 0]$

We have that

$$E[X|X \neq 0] = \sum_{i=1}^4 i P[X=i|X \neq 0]$$

$$= \sum_{i=1}^4 i \frac{P[X=i, X \neq 0]}{P[X \neq 0]}$$

$$= \sum_{i=1}^4 i \frac{P[X=i]}{P[X \neq 0]}$$

We know that $P[X=i] = \pi_i$ ($i=0,1,\dots,4$)

and $P[X \neq 0] = \pi_1 + \pi_2 + \pi_3 + \pi_4$

Therefore,

$$E[X|X \neq 0] = \sum_{i=1}^4 i \frac{\pi_i}{\pi_1 + \pi_2 + \pi_3 + \pi_4}$$

$$= 2.4239.$$

Problem 5.19

Let's choose $\gamma = \gamma_1 + \gamma_2$. Then:

$$P_{01}^U = \frac{\gamma}{\gamma}, \quad P_{00}^U = 1 - \frac{\gamma}{\gamma}$$

$$P_{12}^U = \frac{\gamma}{\gamma}, \quad P_{13}^U = \frac{\mu_1}{\gamma}, \quad P_{11}^U = \frac{\mu_2}{\gamma}$$

$$P_{24}^U = \frac{\mu_1}{\gamma}, \quad P_{22}^U = 1 - \frac{\mu_1}{\gamma}$$

$$P_{30}^U = \frac{\mu_2}{\gamma}, \quad P_{34}^U = \frac{\gamma}{\gamma}, \quad P_{33}^U = \frac{\mu_1}{\gamma}$$

$$P_{42}^U = \frac{\mu_2}{\gamma}, \quad P_{44}^U = \frac{\gamma + \mu_1}{\gamma}$$

With $\gamma = 1, \mu_1 = \frac{3}{2}, \mu_2 = \frac{7}{4}$ we get the following transition probability matrix:

$$P = \begin{bmatrix} 13/17 & 4/17 & 0 & 0 & 0 \\ 0 & 7/17 & 4/17 & 6/17 & 0 \\ 0 & 0 & 11/17 & 0 & 6/17 \\ 7/17 & 0 & 0 & 6/17 & 4/17 \\ 0 & 7/17 & 0 & 0 & 10/17 \end{bmatrix}$$

Solving the following system of linear equations:

$$[\pi_0 \ \pi_1 \ \pi_2 \ \pi_3 \ \pi_4] = [\pi_0 \ \pi_1 \ \pi_2 \ \pi_3 \ \pi_4] P$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

we get that

$$\pi_0 = 0.2357$$

$$\pi_1 = 0.2469$$

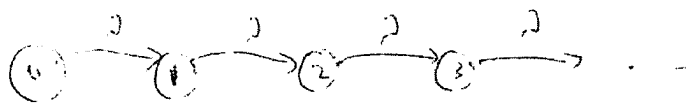
$$\pi_2 = 0.1646$$

$$\pi_3 = 0.1347$$

$$\pi_4 = 0.2181$$

which are the same as in part (b) of problem 5.16.

b) This process can be depicted as follows



Clearly, $p_{ij}(t) = 0$, $\forall i, j$ with $j < i$.

$$\begin{aligned} \text{For } j \geq i, \quad p_{ij}(t) &= \text{Prob} [X(t) = j \mid X(0) = i] = \\ &= \text{Prob} [j-i \text{ state transitions in time interval of length } t] \end{aligned}$$

But all these state transitions occur according to a Poisson process with rate λ , and therefore, for $j \geq i$:

$$p_{ij}(t) = \frac{(\lambda t)^{j-i}}{(j-i)!} e^{-\lambda t}$$

Next, we provide also a proof of this result based on uniformization:

~~For~~ this CTMC, we can pick the uniformizing rate ν equal to λ , which implies the following structure for the transition probabilities for the embedded DTMC:

$$p_{ij}^* = \begin{cases} 1, & j = i+1 \\ 0, & \text{o.w.} \end{cases} \quad (1)$$

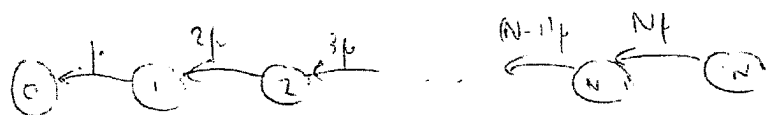
Then, according to the relevant theory presented in class:

$$p_{ij}(t) = \sum_{n=0}^{\infty} p_{ij}^{*(n)} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad (2)$$

but because of (1) above, $p_{ij}^{*(n)} = \begin{cases} 1, & \text{for } n = j-i \\ 0, & \text{o.w.} \end{cases}$ and therefore,

$$p_{ij}(t) = e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!}$$

c) The considered ~~process~~ process can be depicted as follows:



and we want to show that

$$P_i(t) \equiv P[X(t)=i | X(0)=N] = \binom{N}{i} (e^{-t})^i (1 - e^{-t})^{N-i}$$

The considered process can be interpreted as the process that traces the deaths of N individuals, where each individual is dying with instantaneous rate t , and this event takes place independent of what is happening to the other individuals.

Let the lifespan of any of these individuals, be denoted by T . Then

~~According~~ according to the above, T follows an exponential distribution with rate p . Hence, the probability that any given individual survives by time t is equal to $P[T \geq t] = e^{-t}$.

But then,

$$\begin{aligned} P_i(t) &= P[X(t)=i | X(0)=N] = \text{h.p.} [i \text{ individuals survive by time } t] \\ &= \cancel{\text{binomial}(N, e^{-t})} \binom{N}{i} (e^{-t})^i (1 - e^{-t})^{N-i} \end{aligned}$$

as suggested by the problem.

Remark. An ~~analytical characterization~~ ^{alternative derivation} of this result, on the

basis of the computation of the transition probabilities of the considered

(TMC, is provided in the next page. (This is taken from the solution manual of your textbook, so the applying notation and Eq. number, are in reference to that material).

Problem 7.18

By substituting $\lambda_j = 0$ for all $j = 0, 1, \dots$ and $\mu_j = j\mu$, $j = 1, 2, \dots, N$, in (7.98) and (7.99) we get:

$$\frac{d\pi_N(t)}{dt} = -N\mu\pi_N(t) \quad j=N$$

$$\frac{d\pi_j(t)}{dt} = -j\mu\pi_j(t) + (j+1)\mu\pi_{j+1}(t) \quad 0 < j < N$$

$$\frac{d\pi_0(t)}{dt} = \mu\pi_1(t) \quad j=0$$

Solving the first of the above equations we get:

$$\left. \begin{aligned} \frac{d\pi_N(t)}{dt} &= -N\mu\pi_N(t) \Rightarrow \pi_N(t) = C e^{-N\mu t} \\ \pi_N(0) &= 1 \end{aligned} \right\} \Rightarrow C=1 \Rightarrow \pi_N(t) = e^{-N\mu t}$$

For $j=N-1$, we have:

$$\frac{d\pi_{N-1}(t)}{dt} = -(N-1)\mu\pi_{N-1}(t) + N\mu\pi_N(t) \Rightarrow \frac{d\pi_{N-1}(t)}{dt} + (N-1)\mu\pi_{N-1}(t) = N\mu\pi_N(t) \quad (*)$$

A solution of the homogeneous equation $\frac{d\pi_{N-1}(t)}{dt} + (N-1)\mu\pi_{N-1}(t) = 0$ is: $\pi_{N-1}^h(t) = C e^{-(N-1)\mu t}$

A particular solution of (*) is: $\pi_{N-1}^p(t) = -N e^{-N\mu t}$

Thus, $\pi_{N-1}(t) = \pi_{N-1}^h(t) + \pi_{N-1}^p(t)$

$$= C e^{-(N-1)\mu t} - N e^{-N\mu t}$$

But $\pi_{N-1}(0) = 0$ implies that $C=N$. Hence,

$$\pi_{N-1}(t) = N e^{-(N-1)\mu t} - N e^{-N\mu t} = N e^{-(N-1)\mu t} (1 - e^{-\mu t})$$

$$\Rightarrow \pi_{N-1}(t) = \binom{N}{N-1} (e^{-\mu t})^{N-1} (1 - e^{-\mu t})^{N-(N-1)}$$

Proceeding as above and using induction it can be proved that

$$\pi_j(t) = \binom{N}{j} (e^{-\mu t})^j (1 - e^{-\mu t})^{N-j}$$

i.e. $\pi_j(t)$ has a polynomial distribution with parameters N and $e^{-\mu t}$

d) Having $X(t)$ defined as suggested in the problem, it is clear that the untimed behavior of the resulting process will be described by the following transition diagram:



Hence, to argue that the stochastic process $\{X(t), t \geq 0\}$ is a CTMC, it suffices to show that the sojourn time at any given state x , $x \in \{k, \dots, N\}$ is exponentially distributed with a rate that depends just upon this state.

But from the problem description, it is also clear that the counting process that traces the infecting contacts (and especially the first such contact) in any state x , is Poisson with rate

$$\lambda_x = \lambda \cdot \text{Prob}(\text{infecting contact} \mid x \text{ infections}) \quad (1)$$

Hence, the time until the first infection at state x is exponentially distributed with rate λ_x and expected value $\tau_x = 1/\lambda_x$.

From the above transition diagram we can also see that

$$E[\text{time all population is infected}] = \sum_{x=k}^{N-1} \tau_x = \sum_{x=k}^{N-1} 1/\lambda_x. \quad (2)$$

It remains to characterize λ_x , which, from (1), reduces to characterizing the $\text{Prob}(\text{infecting contact} \mid x \text{ infections})$.

This last quantity can be characterized as follows:

$$\text{Prob}(\text{infecting contact} \mid x \text{ infections}) = \frac{\# \text{ of potentially inf. contacts}}{\# \text{ of all possible contacts}} \cdot p \quad (3)$$

(2)

Obviously: # of all possible contacts = $\binom{N}{2}$ (4)

Also, # of potentially inf. contacts = $\binom{N}{2} - \binom{x}{2} - \binom{N-x}{2} =$

$$= \frac{N!}{2!(N-2)!} - \frac{x!}{2!(x-2)!} - \frac{(N-x)!}{2!(N-x-2)!} =$$

$$= \frac{1}{2} [N^2 - N - x^2 + x - (N-x)^2 + (N-x)] = \dots$$

$$\dots = \frac{1}{2} [-2x^2 + 2Nx] = x(N-x) \quad (5)$$

From (1), (3), (4) and (5):

$$d_x = \lambda \frac{x(N-x)}{\binom{N}{2}} p \quad \text{and} \quad \tau_x = \frac{\binom{N}{2}}{\lambda p x(N-x)} \quad (6)$$

Finally, from (2) and (6)

$$E[\text{time all population is infected}] = \frac{N(N-1)}{2\lambda p} \sum_{x=1}^{N-1} \frac{1}{x(N-x)}$$

e) This problem is an instance of the "coupon collection" problem that is discussed in the following pages extracted from Ross' book "Introduction to Probability Models" (10th ed.).

While the solution mentioned above highlights the advantage that can be obtained by modelling the underlying sampling process as a Poisson process and further exploiting the properties of the latter, I am also discussing next an alternative solution. ~~that has been motivated~~

~~by some remarks that came up while grading your homework~~
~~(in essence, this solution tries to give full expression to some "half-baked" thoughts that appeared in some homeworks).~~

Letting r.v. N denote the required # of samples, we have:

$$\begin{aligned} E[N] &= \sum_{i=1}^m E[\text{# of samples to get the } i\text{-th new sample}] = \\ &= \sum_{i=1}^m E_x \left[E[\text{# of samples to get the } i\text{-th new sample} \mid \text{set of currently collected samples} = X] \right] \\ &= \sum_{i=1}^m \sum_{X \in \mathcal{X}^{i-1}} E[\text{# of samples to get the } i\text{-th new sample} \mid X] \cdot p(X) \end{aligned}$$

where $\mathcal{X}^{i-1}$ is all the subsets of $\{1, 2, \dots, m\}$ with cardinality $i-1$

and $p(X \in \mathcal{X}^i) = \text{Prob} \{ \text{the first } i \text{ collected samples are those in } X \}$.

Then

$$E[\text{# of samples to get the } i\text{-th new sample} \mid X] = \frac{1}{\sum_{j \in X^c} p_j}$$

To get the $P(X)$, we work as follows:

~~for $X \in X^0$~~ Obviously, $X \in X^0 = \emptyset$ and $P(\emptyset) = 1$.

for $X \in X^1$, $X = \{i\}$ for some $i \in \{1, 2, \dots, m\}$, and

~~the case~~ $P(\{i\}) = p_i$

Now, consider the case $X \in X^2$, i.e., $X = \{i, j\}$ with $i \neq j$.

Then,

$$\begin{aligned} P(\{i, j\}) &= P(\{i, j\} | \{i\}) P(\{i\}) + P(\{i, j\} | \{j\}) P(\{j\}) = \\ &= \frac{p_j}{1 - p_i} p_i + \frac{p_i}{1 - p_j} p_j \end{aligned}$$

Working in this way, we see that we have the more general recursion: $\forall i = 1, \dots, m, \forall X \in X^i$,

~~for $X \in X^i$~~

$$P(X) = \sum_{k \in X} P(X \setminus \{k\}) P(X \setminus \{k\}) =$$

$$= \sum_{k \in X} \frac{p_k}{1 - \sum_{l \in X \setminus \{k\}} p_l} P(X \setminus \{k\})$$

the case where the classification is into any one of r possible groups, we have the following application to a model of employees moving about in an organization.

Example 5.16 Consider a system in which individuals at any time are classified as being in one of r possible states, and assume that an individual changes states in accordance with a Markov chain having transition probabilities P_{ij} , $i, j = 1, \dots, r$. That is, if an individual is in state i during a time period then, independently of its previous states, it will be in state j during the next time period with probability P_{ij} . The individuals are assumed to move through the system independently of each other. Suppose that the numbers of people initially in states $1, 2, \dots, r$ are independent Poisson random variables with respective means $\lambda_1, \lambda_2, \dots, \lambda_r$. We are interested in determining the joint distribution of the numbers of individuals in states $1, 2, \dots, r$ at some time n .

Solution: For fixed i , let $N_j(i)$, $j = 1, \dots, r$ denote the number of those individuals, initially in state i , that are in state j at time n . Now each of the (Poisson distributed) number of people initially in state i will, independently of each other, be in state j at time n with probability P_{ij}^n , where P_{ij}^n is the n -stage transition probability for the Markov chain having transition probabilities P_{ij} . Hence, the $N_j(i)$, $j = 1, \dots, r$ will be independent Poisson random variables with respective means $\lambda_i P_{ij}^n$, $j = 1, \dots, r$. Because the sum of independent Poisson random variables is itself a Poisson random variable, it follows that the number of individuals in state j at time n —namely $\sum_{i=1}^r N_j(i)$ —will be independent Poisson random variables with respective means $\sum_{i=1}^r \lambda_i P_{ij}^n$, for $j = 1, \dots, r$. ■

Example 5.17 (The Coupon Collecting Problem) There are m different types of coupons. Each time a person collects a coupon it is, independently of ones previously obtained, a type j coupon with probability p_j , $\sum_{j=1}^m p_j = 1$. Let N denote the number of coupons one needs to collect in order to have a complete collection of at least one of each type. Find $E[N]$.

Solution: If we let N_j denote the number one must collect to obtain a type j coupon, then we can express N as

$$N = \max_{1 \leq j \leq m} N_j$$

However, even though each N_j is geometric with parameter p_j , the foregoing representation of N is not that useful, because the random variables N_j are not independent.

We can, however, transform the problem into one of determining the expected value of the maximum of *independent* random variables. To do so, suppose that coupons are collected at times chosen according to a Poisson process with rate $\lambda = 1$. Say that an event of this Poisson process is of type j , $1 \leq j \leq m$, if the coupon obtained at that time is a type j coupon. If we now let $N_j(t)$ denote the number of type j coupons collected by time t , then it follows

from Proposition 5.2 that $\{N_j(t), t \geq 0\}$, $j = 1, \dots, m$ are independent Poisson processes with respective rates $\lambda p_j = p_j$. Let X_j denote the time of the first event of the j th process, and let

$$X = \max_{1 \leq j \leq m} X_j$$

denote the time at which a complete collection is amassed. Since the X_j are independent exponential random variables with respective rates p_j , it follows that

$$\begin{aligned} P\{X < t\} &= P\{\max_{1 \leq j \leq m} X_j < t\} \\ &= P\{X_j < t, \text{ for } j = 1, \dots, m\} \\ &= \prod_{j=1}^m (1 - e^{-p_j t}) \end{aligned}$$

Therefore,

$$\begin{aligned} E[X] &= \int_0^\infty P\{X > t\} dt \\ &= \int_0^\infty \left\{ 1 - \prod_{j=1}^m (1 - e^{-p_j t}) \right\} dt \end{aligned} \quad (5.15)$$

It remains to relate $E[X]$, the expected time until one has a complete set, to $E[N]$, the expected number of coupons it takes. This can be done by letting T_i denote the i th interarrival time of the Poisson process that counts the number of coupons obtained. Then it is easy to see that

$$X = \sum_{i=1}^N T_i$$

Since the T_i are independent exponentials with rate 1, and N is independent of the T_i , we see that

$$E[X|N] = NE[T_i] = N$$

Therefore,

$$E[X] = E[N]$$

and so $E[N]$ is as given in Equation (5.15).

f) Let

- r.v. T = time between your arrival at the station and the last train departure.

- r.v. X = # of customers you meet when you arrive

- t = the time between two consecutive train arrivals

The key insight for answering this question is the result established in one of the problems in Hw #2, that the occurrence time of an event from a Poisson process that takes place in a certain time interval is uniformly distributed in that interval. Hence,

$$\begin{aligned} E[X] &= E[E[X|T]] = E[\lambda T] = \lambda E[T] = \\ &= \lambda t/2 = 1 \cdot \frac{5}{2} = 10. \end{aligned}$$