Solutions for HW2

6.5!

a) Let T denote the interarrival time. Then T is exponentially distributed with rate λ. Therefore

$$P(T > \tau) = 1 - P(T < \tau) = 1 - (1 - e^{-\lambda\tau/60}) = e^{-\lambda\tau/60}$$

(The division by 60 is necessary since λ is expressed in jobs/hr and τ in minutes).

b) Let $N(\tau)$ denote the number of jobs arriving over an interval of length τ. Then, we are interested in the probability of the event.

$$N(a) = 0 \wedge N(60-b, 60) = 0$$

Since the arrival process is Poisson it has independent and stationary increments. Therefore,

$$P(N(a) = 0 \wedge N(60-b, 60) = 0) = P(N(a) = 0) \cdot P(N(b) = 0) =$$

$$= e^{-\lambda a/60} e^{-\lambda b/60} = e^{-\lambda(a+b)/60}$$

where the division by 60 is performed for the same reason as above.

c) A compact expression for this probability is obtained by noticing that the complement of the considered event is that at most one job is submitted within the first a minutes. Hence,

$$P[N(a) \geq 2] = 1 - P[N(a) \leq 1] = 1 - e^{-\lambda a/60} - \frac{\lambda a}{60} e^{-\lambda a/60} =$$

$$= 1 - \left(1 + \frac{\lambda a}{60}\right) e^{-\lambda a/60}$$

6.6) Let E_n denote the event that exactly one event occurs in each of the intervals $(s_i, t_i]$, $i = 1, \ldots, n$. Then,

$$P[E_n \mid \text{exactly } n \text{ events occur in } (0,t]] =$$

$$= \frac{P[E_n \wedge \text{exactly } n \text{ events occur in } (0,t]]}{P[\text{exactly } n \text{ events occur in } (0,t)]} =$$

$$= \frac{P[N(0,s_1) = 0 \wedge N(s_1,t_1) = 1 \wedge N(t_1,s_2) = 0 \wedge \ldots \wedge N(s_n,t_n) = 1 \wedge N(t_n,t) = 0]}{\frac{(\lambda t)^n}{n!} e^{-\lambda t}} =$$

$$= \frac{e^{-\lambda s_1}\left(\lambda \tau_1 e^{-\lambda \tau_1}\right) e^{-\lambda(s_2 - t_1)} \ldots \left(\lambda \tau_n e^{-\lambda(t_n - s_n)}\right) e^{-\lambda(t - t_n)}}{\frac{(\lambda t)^n}{n!} e^{-\lambda t}} =$$

$$= \frac{\lambda^n \tau_1 \tau_2 \ldots \tau_n \; e^{-\lambda[s_1 + t_1 - s_1 + \ldots + t_n - s_n + t - t_n]}}{\frac{(\lambda t)^n}{n!} e^{-\lambda t}} =$$

$$n! \frac{\left(\prod_{i=1}^{n} \tau_i\right) e^{-\lambda t}}{t^n e^{-\lambda t}} = n! \prod_{i=1}^{n} \left(\frac{\tau_i}{t}\right)$$

<u>Remark 1</u>: The above result when considered for $n = 1$ implies that the distribution of the occurrence time of a <u>single</u> event that took place by time t according to a Poisson arrival process, is uniform over the interval $(0, t]$.

6.11) Let X denote the number of customers that arrive during a service time and S be the random length of the service time. Then,

$$P[X=k] = \int_0^\infty P[X=k \mid S=t]\, f_S(t)\, dt$$

where $f_S(t)$ is the density function of S, given by

$$f_S(t) = \mu e^{-\mu t}, \quad t \geq 0.$$

Also we have,

$$P[X=k \mid S=t] = P[k \text{ customers arrive during the interval } [0,t]] =$$

$$= \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t \geq 0, \quad k = 0,1,2,\ldots$$

Therefore,

$$P[X=k] = \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} \mu e^{-\mu t}\, dt = \frac{\mu \lambda^k}{k!} \int_0^\infty t^k e^{-(\lambda+\mu)t}\, dt$$

For $k=0$:

$$P[X=0] = \mu \int_0^\infty e^{-(\lambda+\mu)t}\, dt = \frac{\mu}{\lambda+\mu} \quad \text{(which should look familiar!)}$$

For $k \geq 1$.

$$P[X=k] = \frac{\mu \lambda^k}{k!} \int_0^\infty \frac{[(\lambda+\mu)t]^k}{(\lambda+\mu)^k} e^{-(\lambda+\mu)t} \frac{d[(\lambda+\mu)t]}{\lambda+\mu} =$$

$$= \frac{\mu \lambda^k}{k!} \frac{1}{(\lambda+\mu)^{k+1}} \int_0^\infty [(\lambda+\mu)t]^k e^{-(\lambda+\mu)t}\, d[(\lambda+\mu)t] =$$

$$= \frac{\mu \lambda^k}{\cancel{k!}} \frac{1}{(\lambda+\mu)^{k+1}} \cancel{\Gamma(k+1)} = \frac{\mu \lambda^k}{(\lambda+\mu)^{k+1}}$$

where in the next to last equality we have used the results provided in the hint.

<u>Remark</u>. The last result is also quite intuitive. To understand it, notice that the event $\{X = k\}$ is equivalent to the event that the (Poisson) counting process that combines arrivals and departures must observe $k+1$ events, of which the first k must be arrivals. Hence, the corresponding probability is

$$\left(\frac{\lambda}{\lambda+\mu}\right)^k \frac{\mu}{\lambda+\mu}.$$

b) The conditional distribution of X, given that $X > 1$, is the same with the unconditional distribution of $1 + Y$, where Y is the residual life of X after the first unit. But since X is exponentially distributed, Y has the same statistics with X itself. Hence, the correct answer is **1**.

c) Let T_i denote the time between the $(i-1)$st and the i-th failure. Then the r.v.'s T_i are independent and exponentially distributed, each with rate $(101-i)/200$ hrs^{-1}. Therefore

$$E[T] = E[T_1 + T_2 + T_3 + T_4 + T_5] = \sum_{i=1}^{5} E[T_i] =$$

$$= \sum_{i=1}^{5} \frac{200}{101-i} .$$

Similarly

$$Var[T] = \sum_{i=1}^{5} Var[T_i] = \sum_{i=1}^{5} \frac{200^2}{(101-i)^2}$$

d) (1) The considered event is equivalent to the event that your processing time in ~~server~~ server 1 will be smaller than the remaining processing time of A in server 2. Since these two random variables are exponentially distributed with corresponding rates μ_1 and μ_2, the sought probability is equal to $\mu_1/(\mu_1+\mu_2)$

(2) The complementary event to the event considered in this question is that we will have two service completions at server 2 before a service completion at server 1. Thinking as in the interpretation of the result of problem 6.11 in your textbook (see above), the prob of this last event is equal to $\left(\frac{\mu_2}{\mu_1+\mu_2}\right)^2$. Hence, the prob. of the considered event is equal to $1 - \left(\frac{\mu_2}{\mu_1+\mu_2}\right)^2$

(8)

d) (3) One way to answer this question is by noticing that

$$E[T] = E[\text{Your own total processing time}] +$$
$$P_A \cdot E[\text{remaining time of A in server 2 when you arrive at station 2}] +$$
$$P_B \cdot E[\text{remaining time of B in server 2 when you arrive at station 2}] =$$

$$= \frac{1}{\mu_1} + \frac{1}{\mu_2} + P_A \cdot \frac{1}{\mu_2} + P_B \cdot \frac{1}{\mu_2}$$

where P_A and P_B are taken from the previous parts of this problem.

e) 1) A will obtain a new kidney simply if one arrives before he dies. This will occur with probability $\dfrac{\lambda}{\lambda + \mu_A}$

2) B will obtain a new kidney if A obtains a new kidney or dies while he is still alive, and subsequently a kidney arrives before B himself dies. Therefore, the corresponding probability is

$$\left(\frac{\lambda + \mu_A}{\lambda + \mu_A + \mu_B}\right)\left(\frac{\lambda}{\lambda + \mu_B}\right)$$

f)- ~~[illegible]~~

$$E[S] = \sum_{i=1}^{n} E[T_i] = \sum_{i=1}^{n} 1/\lambda = n/\lambda$$

and

$$Var[S] = \sum_{i=1}^{n} Var[T_i] = \sum_{i=1}^{n} 1/\lambda^2 = n/\lambda^2$$

Also,

$$SCV[S] = \frac{Var[S]}{E[S]^2} = \frac{n/\lambda^2}{(n/\lambda)^2} = \frac{1}{n}$$

Finally, to compute the pdf and cdf of r.v. S, we work as follows:

First we establish the pdf formula by induction:

For $n=1$, the provided formula becomes.

$$f(s) = \begin{cases} \lambda e^{-\lambda s}, & \text{if } s>0 \\ 0, & \text{o.w.} \end{cases}$$

which is correct since in that case $S = T_1$, i.e., exponentially distributed with rate λ.

Then suppose that the formula holds true for $n \le k$. To prove it true for $n=k+1$, notice that the convolution integral which provides the cdf of the sum of two continuous independent r.v.'s (as discussed in class), implies that.

$$f_{T_1+T_2+\dots+T_k+T_{k+1}}(s) = \int_0^\infty f_{T_{k+1}}(s-y)\, f_{T_1+T_2+\dots+T_k}(y)\, dy =$$

$$= \int_0^s \lambda e^{-\lambda(s-y)} \lambda e^{-\lambda y} \frac{(\lambda y)^{k-1}}{(k-1)!} dy = \frac{\lambda e^{-\lambda s}}{(k-1)!} \int_0^s \frac{(\lambda y)^{k-1}}{[illegible]} d(\lambda y) = \frac{\lambda e^{-\lambda s}}{(k-1)!} \frac{(\lambda s)^k}{k} =$$

$$= \frac{\lambda^{k+1} s^k e^{-\lambda s}}{k!}$$

Finally, $F(s)$ can be obtained from $f(s)$ by

$$F(s) = \int_{-\infty}^{s} f(s)\, ds,$$

and integration by parts (formally, you can establish the result by induction).

An alternative way for getting $F(s)$ and $f(s)$:

$$1 - F(s) = P(S > s) = P(\text{at most } n-1 \text{ arrivals in } [0,s]) = $$

$$= \sum_{j=0}^{n-1} \frac{e^{-\lambda s}(\lambda s)^j}{j!} \rightsquigarrow$$

$$\rightsquigarrow \quad F(s) = 1 - \sum_{j=0}^{n-1} \frac{e^{-\lambda s}(\lambda s)^j}{j!}$$

Then

$$f(s) = \frac{dF(s)}{ds} = \lambda e^{-\lambda s} \sum_{j=0}^{n-1} \frac{(\lambda s)^j}{j} - e^{-\lambda s} \sum_{j=1}^{n-1} \lambda \frac{(\lambda s)^{j-1}}{(j-1)!} =$$

$$= \lambda e^{-\lambda s} \left[\sum_{j=0}^{n-1} \frac{(\lambda s)^j}{j!} - \sum_{j=0}^{n-2} \frac{(\lambda s)^j}{j!} \right] =$$

$$= \begin{cases} \dfrac{\lambda^n s^{n-1} e^{-\lambda s}}{(n-1)!} & \text{if } s \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

Another approach could be based on MGFs.

~~10~~

~~**Problem 4 (25 points):** Consider two *independent* exponentially distributed random variables X_1 and X_2 with corresponding rates μ_1 and μ_2. Compute $E[X_2 | X_2 > X_1]$.~~

h) Let $F_{X_2 > X_1}(x) = P[X_2 \le x \mid X_2 > X_1] =$

$$= \frac{P[X_1 < X_2 \le x]}{P[X_2 > X_1]} =$$

$$= \frac{\mu_1 + \mu_2}{\mu_1} \cdot \int_0^x P[X_1 < X_2 \le x \mid X_2 = y]\, dP(X_2 = y) =$$

$$= \frac{\mu_1 + \mu_2}{\mu_1} \int_0^x P[X_1 < y]\, f_{X_2}(y)\, dy$$

where $f_{X_2}(y)$ is the pdf of X_2 (exponential dist. with rate μ_2)

Then,

$$f_{X_2 > X_1}(x) = \frac{d F_{X_2 > X_1}(x)}{dx} = \frac{\mu_1 + \mu_2}{\mu_1} P[X_1 < x]\, f_{X_2}(x) =$$

$$= \frac{\mu_1 + \mu_2}{\mu_1} (1 - e^{-\mu_1 x})\, \mu_2 e^{-\mu_2 x}$$

and

$$E[X_2 \mid X_2 > X_1] = \int_0^\infty x\, f_{X_2 > X_1}(x)\, dx = \frac{\mu_2(\mu_1 + \mu_2)}{\mu_1} \int_0^\infty x\, e^{-\mu_2 x} (1 - e^{\mu_1 x})\, dx$$

$$= \frac{\mu_1 + \mu_2}{\mu_1} \int_0^\infty x\, \mu_2 e^{-\mu_2 x}\, dx - \frac{\mu_2}{\mu_1} \int_0^\infty x (\mu_1 + \mu_2) e^{-(\mu_1 + \mu_2) x}\, dx$$

$$= \frac{\mu_1 + \mu_2}{\mu_1} \frac{1}{\mu_2} - \frac{\mu_2}{\mu_1} \frac{1}{\mu_1 + \mu_2} = \frac{1}{\mu_2} + \frac{\mu_2}{\mu_1} \left(\frac{1}{\mu_2} - \frac{1}{\mu_1 + \mu_2} \right) =$$

$$= \frac{1}{\mu_2} + \frac{1}{\mu_1} \frac{\mu_1}{\mu_1+\mu_2} = \frac{1}{\mu_2} + \frac{1}{\mu_2+\mu_1}$$

Notice that the above result makes perfect sense, since the second term (i.e., $\frac{1}{\mu_2+\mu_1}$) is the expected time until X_1 expires (this is an "exponential race" between the two exponentials) and the first term (i.e., $\frac{1}{\mu_2}$) is the expected remaining time to the expiration of X_2 (due to its memoryless nature). So, with these insights, the above result could have also been derived while arguing from the fundamental properties of the exponential distribution.