Solutions For HW1

6.1)

S_k, $k = 0, 1, 2, \ldots$: no. of messages successfully reaching a receiver

$P(S_k = n) = 1/N$, $n = 1, \ldots, N$, $\forall k$

$X_{k+1} = X_k + S_{k+1}$, $S_0 = 0$, $X_0 = 0$ → This is a more meaningful recursion for the definition of X_k

i.e., X_k = total number of messages successfully received by time k

(a) $\{X_k\}$ does possess the Markov property. ~~The one-step transition~~ At state X_k, the process transitions to states $X_k+1, X_k+2, \ldots, X_k+N$, with probabilities $1/N$, while its probability to transition to any other state from X_k is equal to zero. So, this is a homogeneous, DTMC.

(b) $E[X_k] = \sum_{i=1}^{k} E[S_i]$

$\forall i,\ E[S_i] = E[S_1] = \dfrac{1+2+\ldots+N}{N} = \dfrac{N(N+1)}{N \cdot 2} = \dfrac{N+1}{2}$ $\Rightarrow$

$\Rightarrow E[X_k] = \dfrac{k(N+1)}{2}$

$Var[X_k] = \sum_{i=1}^{k} Var[S_i]$ since the S_i's are independent r.v.'s.

$\forall i,\ Var[S_i] = Var[S_1] = E[S_1^2] - E[S_1]^2$

$E[S_1^2] = \dfrac{1+2^2+\ldots+N^2}{N} = \dfrac{N(N+1)(2N+1)}{N\,6} = \dfrac{(N+1)(2N+1)}{6}$ $\Rightarrow$

$\Rightarrow Var[S_1] = \dfrac{(N+1)(2N+1)}{6} - \dfrac{(N+1)^2}{4} = \dfrac{(N+1)[2(2N+1) - 3(N+1)]}{12}$

$= \dfrac{(N+1)(N-1)}{12} = \dfrac{N^2-1}{12}$

$\Rightarrow Var[X_k] = \dfrac{k(N^2-1)}{12}$

7.3) This is a non-homogeneous MC.

For $(k<i) \vee [k \geq i \wedge (j<i) \vee (j>i+1)]$:

$$P[X_{k+1}=j \mid X_k=i]=0$$

for $j=i+1 \wedge k \geq i$

$$P[X_{k+1}=j \mid X_k=i]=\frac{26-i}{52-k}$$

since by time k, the pack has $52-k$ cards, out of which $26-i$ are red and the rest are black, and the next card is selected randomly.

finally, for $j=i \wedge k \geq i$:

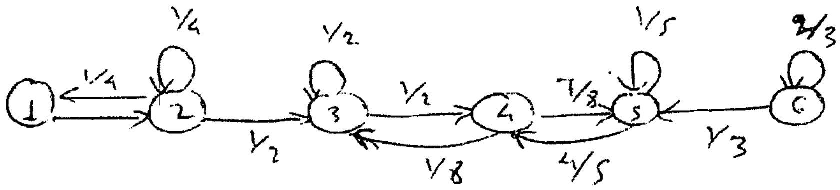
$$P[X_{k+1}=j \mid X_k=i]=1-\frac{26-i}{52-k}$$

Based on the above analysis,

$$P[X_7=6 \mid X_6=5]=\frac{26-5}{52-6}=\frac{21}{46}$$

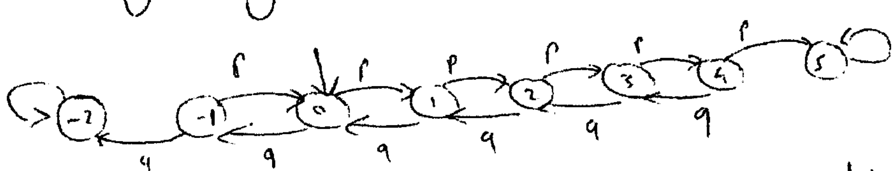
and $$P[X_9=4 \mid X_8=2]=\emptyset.$$

7.7) The most straightforward way to solve this problem is by developing the state transition diagram (STD) of this process. This is as follows:



The states appearing in the above diagram correspond to the natural numbering of the rows and columns of the provided matrix P. It is easy to see from the above diagram that states 1, 2, and 6 are transient while state 3, 4 and 5 constitute the only closed communicating class of the chain. The self-loops at states 2, 3, 5 and 6 imply that all the communicating classes of this chain are aperiodic.

7.11) The dynamics of this game can be represented by the following DTMC, the state of which defines the gambler's gains (losses are represented as negative gains):



$p = \frac{4}{36} = \frac{1}{9}$, since $\exists$ 4 outcomes that can give a five $\{(2,3),(3,2),(4,1),(1,4)\}$ out of 36 (6×6) possible outcomes.

$q = 1 - p = 8/9$

Let X_i denote the (conditional) probability of absorption to state 5, given that the process starts at the transient state i, $i = -1, 0, 1, 2, 3, 4$.

Then by conditioning on the first transition we get:

$$X_0 = pX_1 + qX_{-1}$$
$$X_1 = pX_2 + qX_0$$
$$X_2 = pX_3 + qX_1$$
$$X_3 = pX_4 + qX_2$$
$$X_4 = p + qX_3$$
$$X_{-1} = pX_0$$

This is a system of 6 linear equations in 6 unknowns, and as we discussed in class, it has a unique solution. An explicit way for the solution of this system of equations is provided in the next page [redacted]

7.12)

(a) $P_{ij}^n = P[X_n = j \mid X_0 = i] = \sum_{m=1}^{n} P[X_n = j \mid X_0 = i, T_{ij} = m]$.

$$P[T_{ij} = m \mid X_0 = i] = \sum_{m=1}^{n} P[X_n = j \mid X_m = j]\, P[T_{ij} = m \mid X_0 = i]$$

$$= \sum_{m=1}^{n} P[T_{ij} = m \mid X_0 = i]\; P_{jj}^{n-m}$$

(b) $P_{ij}^n = P[X_n = j \mid X_0 = i] \overset{\text{from (a)}}{=} \sum_{m=1}^{n} P[T_{ij} = m \mid X_0 = i]\, \underbrace{P_{jj}^{n-m}}_{=1}$

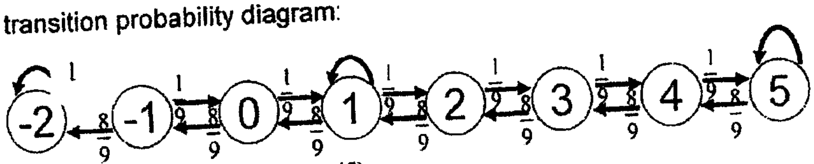
$$= \sum_{m=1}^{n} P[T_{ij} = m \mid X_0 = i] = P[T_{ij} \le n \mid X_0 = i]$$

Problem 7.11

(a.) First, we determine the probability to toss two dice with a total of five

$$p=\frac{4}{36}=\frac{1}{9}$$

Then, we can model this problem to gambler's ruin problem as following transition probability diagram:



And, $X_0=0$, want to solve $\rho_0(5)$

(b.) $\rho_{-2}(5)=0,\ \rho_{5,5}=1$

$$\rho_{-1}(5)=\frac{8}{9}*\rho_{-2}(5)+\frac{1}{9}\rho_0(5)\Rightarrow\rho_0(5)-\rho_{-1}(5)=8(\rho_{-1}(5)-\rho_{-2}(5))$$

$$\rho_0(5)=\frac{8}{9}*\rho_{-1}(5)+\frac{1}{9}\rho_1(5)$$

$$\frac{1}{9}(\rho_1(5)-\rho_0(5))=\frac{8}{9}(\rho_0(5)-\rho_{-1}(5))\Rightarrow\rho_1(5)-\rho_0(5)=8(\rho_0(5)-\rho_{-1}(5))$$

symmetrically, we can get following equation

$$\rho_0(5)-\rho_{-1}(5)=8[\rho_{-1}(5)-\rho_{-2}(5)]$$

$$\rho_1(5)-\rho_0(5)=8[\rho_0(5)-\rho_{-1}(5)]=8^2[\rho_{-1}(5)-\rho_{-2}(5)]$$

$$\rho_2(5)-\rho_1(5)=8[\rho_1(5)-\rho_0(5)]=8^3[\rho_{-1}(5)-\rho_{-2}(5)]$$

$$\rho_3(5)-\rho_2(5)=8[\rho_2(5)-\rho_1(5)]=8^4[\rho_{-1}(5)-\rho_{-2}(5)]$$

$$\rho_4(5)-\rho_3(5)=8[\rho_3(5)-\rho_2(5)]=8^5[\rho_{-1}(5)-\rho_{-2}(5)]$$

$$\rho_5(5)-\rho_4(5)=8[\rho_4(5)-\rho_3(5)]=8^6[\rho_{-1}(5)-\rho_{-2}(5)]$$

Then do the summation for both sides

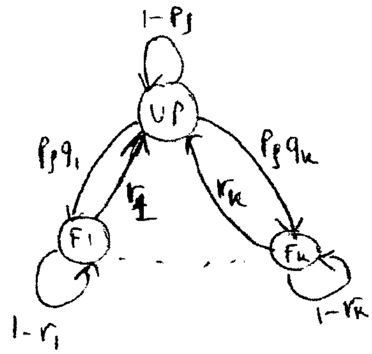
$$\rho_5(5)-\rho_{-1}(5)=299592[\rho_{-1}(5)-\rho_{-2}(5)]$$

$$1-\rho_{-1}(5)=299592\rho_{-1}(5)\Rightarrow\rho_{-1}(5)=\frac{1}{299593}$$

$$\rho_{-1}(5)=\frac{1}{9}\rho_0(5)\Rightarrow\rho_0(5)=\frac{9}{299593}$$

B)

The operation of this production line can be modeled by the following DTMC:



$$P = \begin{array}{c} \\ UP \\ F_1 \\ F_2 \\ \vdots \\ F_k \end{array} \begin{array}{c} \begin{array}{ccccc} UP & F_1 & F_2 & \cdots & F_k \end{array} \\ \left[\begin{array}{ccccc} 1-p_f & p_f q_1 & p_f q_2 & & p_f q_k \\ r_1 & 1-r_1 & & & \\ r_2 & & 1-r_2 & & \\ \vdots & & & \ddots & \vdots \\ r_k & - & & - & 1-r_k \end{array}\right] \end{array}$$

In the above STD, UP denotes the state where the line is operational and F_i is the state where it fails according to failure mode i, $i=1,\dots,k$. Assuming that all involved probabilities obtain ~~non extreme values~~ positive, the chain is irreducible, recurrent and aperiodic, and therefore it has a limiting distribution. Let π_{UP}, π_i, $i=1,\dots,k$, denote the limiting probabilities. Then, from $\pi^T = \pi^T P$, we get

$$\pi_{UP} = (1-p_f)\pi_{UP} + \sum_j r_j \pi_j$$

$$\forall j, \quad \pi_j = p_f q_j \pi_{UP} + (1-r_j)\pi_j \iff \pi_j = \frac{p_f q_j}{r_j}\pi_{UP}$$

Furthermore, from $\pi_{UP} + \sum_j \pi_j = 1$, we get:

$$\pi_{UP}\left(1 + \sum_j \frac{p_f q_j}{r_j}\right) = 1 \iff$$

$$\iff \pi_{UP} = \frac{1}{1 + p_f \sum \frac{q_j}{r_j}} = \frac{1/p_f}{1/p_f + \sum_j q_j (1/r_j)} = TH$$

The last result is very intuitive since:

- $1/p_f$ is the expected sojourn time for state UP
- $1/r_j$ " " " " " " F_j
- $\sum_j q_j (1/r_j)$ is the expected duration of a failure

c) (i) We have:

$$\sum_{n=0}^{\infty} P_{ii}^{(n)} = \sum_{n=0}^{\infty} P\{X_n = i \mid X_0 = i\} =$$

$$= \sum_{n=0}^{\infty} E\left[I_{\{X_n=i\}} \mid X_0 = i \right] =$$

$$= E\left[\sum_{n=0}^{\infty} I_{\{X_n=i\}} \mid X_0 = i \right]$$

The last expectation is equal to ∞ if and only if state i is recurrent. Also, notice that the exchange of the summation and the expectation, that were performed in the last step, is justified by the positivity of $I_{\{X_n=i\}}$ and the theorem of monotone convergence.

(ii) Since i is recurrent and communicates with j,:

$$\exists\ P_{ij}^{(k)} > 0 \text{ and } P_{ji}^{(m)} > 0 \text{ for some } k \text{ and } m$$

(Remember that $P_{ij}^{(k)}$ denotes the cond. prob. that the process in k steps will be in state j given that it starts at state i)

Notice that we also have:

$$\forall n, \quad P_{jj}^{(m+n+k)} \geq P_{ji}^{(m)} \cdot P_{ii}^{(n)} \cdot P_{ij}^{(k)}$$

since the rhs in the above inequality is only one possibility for obtaining the outcome implied in the lhs.

Hence, $$\sum_{n=1}^{\infty} P_{jj}^{(m+n+k)} \geq P_{ji}^{(m)} P_{ij}^{(k)} \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \quad (1)$$

since $P_{ji}^{(m)}, P_{ij}^{(k)} > 0$ and $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$ from part (i).

Therefore, j is recurrent as well (from (1) and part (i))

(iii) Just notice that if i is transient and communicates with j, and j is recurrent, then i should be recurrent as well (from part (ii)).

D) For part (i), notice that:

$$\underline{f} = (I - P_T)^{-1}\underline{1} \Leftrightarrow (I - P_T)\underline{f} = \underline{1} \Leftrightarrow \underline{f} = \underline{1} + P_T\underline{f} \Leftrightarrow$$

$$\Leftrightarrow \quad \forall i \in \{1,\dots,n\}: \quad f_i = 1 + \sum_{j=1}^{n} P_{ij} f_j \Leftrightarrow$$

$$\Leftrightarrow \quad \forall i \in \{1,\dots,n\}: \quad f_i = 1 \cdot \sum_{j=0}^{n} P_{ij} + \sum_{j=1}^{n} P_{ij} f_j \Leftrightarrow$$

$$\Leftrightarrow \quad \forall i \in \{1,\dots,n\}: \quad f_i = 1 \cdot P_{i0} + \sum_{j=1}^{n} (1 + f_j) P_{ij}$$

But the last statement is true since the provided recursions essentially characterize each f_i, $i = 1,\dots,n$, by conditioning on the first transition out of state i.

Regarding part (ii), a first intuitive interpretation of Eq. (2) is, of course, through the development provided for part (i) above.

We also have that:

$$(I - P_T)\sum_{k=0}^{\infty} P_T^k = \sum_{k=0}^{\infty} P_T^k - \sum_{k=0}^{\infty} P_T^{k+1} = P_T^0 = I \rightsquigarrow$$

$$\rightsquigarrow \quad (I - P_T)^{-1} = \sum_{k=0}^{\infty} P_T^k \qquad Ⓐ$$

Let $(I - P_T)^{-1} \equiv F$ Ⓑ (F is frequently referred to as the "fundamental matrix" of the MC)

From Ⓐ, working as in the solution of part B(i) of this homework, we can infer that:

$F(i,j)$ = expected # of visits to transient state j before absorption, when starting from transient state i.

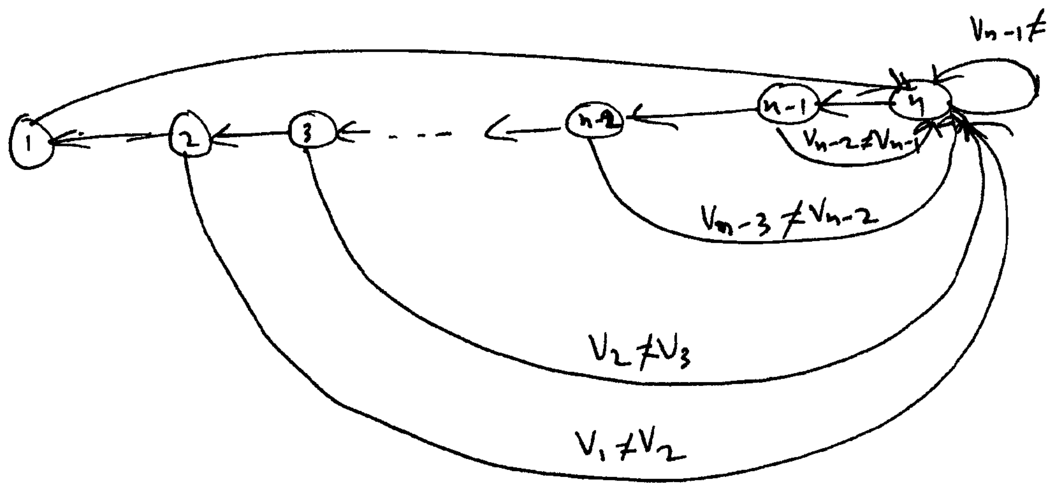
But then, it should be clear that the provided Eq. (2) simply computes f_i, for any transient state i, as the expected # of visits to all transient states before absorption.

e) If the matrix T that was introduced in the proof of the original result, that was discussed in class, remains stochastic and primitive (i.e., stochastic, irreducible and aperiodic), then the line will still converge to the limiting regime that is defined by the theorem presented in class.

Remember that

$$T = \begin{array}{c|cccccc} & 1 & 2 & & (n-2) & (n-1) & n \\ \hline 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 2 & v_1/v_2 & 0 & \cdots & 0 & 0 & 1-v_1/v_2 \\ \vdots & \vdots & \vdots & & \vdots & & \\ n-1 & 0 & 0 & & v_{n-2}/v_{n-1} & 0 & 1-v_{n-2}/v_{n-1} \\ n & 0 & 0 & & 0 & v_{n-1}/v_n & 1-v_{n-1}/v_n \end{array}$$

Also, a graphical representation of T's structure is as follows:



In the above graph, the arcs leading to n will be present only if the corresponding inequality holds (the arc from 1 to n will always be present).

From the above, it can be easily checked that even if $v_i \neq v_{i+1}$ for some i's, T remains stochastic and irreducible.

On the other hand, it can also be checked in the provided graph that T will become periodic if the cardinalities of the subsets that collect all the v_is with equal values have a gcd greater than 1 (this characterization includes trivially the case where $v_1 = v_2 = \ldots = v_n$).

In this last case, starting all the workers at the origin of the line, they will eventually organize into groups of cardinality equal to the aforementioned gcd that will move together as a single worker, carrying, however, the work of gcd workers. This can be formally proved when taking into consideration the dynamics of the handover process in bucket brigades. Once the emergence of this grouping has been established, then the limiting regime can be obtained from the original result presented in class by treating each group as a single worker of the corresponding speed. This regime will define the area over which each group will work (as a team) ~~[illegible]~~ as $t \to \infty$. (and, of course, each cycle will complete ~~[illegible]~~ gcd orders instead of just one!)

One last remark is that, in this new situation, the initial condition matters! To see this, consider a line with only two workers and $v_1 = v_2$. If we start both of them at the beginning of the line, then it is easy to see that they will keep traversing the entire line in sync. If, on the other hand, one of them is started at the beginning of the line and the other in the middle of the line, then, they will split the line into two equal segments and each will work in one of them.

Finally, let me add that the above remarks derive from the mathematical developments presented in class, but they are not completely formal. One could try to fill in the details in the arguments provided above, or alternatively to try to characterize the limiting behavior of the line using the variation(s) of the P-F theorem for periodic matrices. In any case, it is interesting to see how mathematical concepts like the periodicity of a matrix translate into behavioral patterns like the group formation that was described above.