ISyE7201: Production and Service System Engineering Instructor: Spyros Reveliotis Spring 2016

Homework #4

Due Date: March 28, 2016

A. Reading assignment: This homework focuses on the modeling and analysis of single queueing stations, with most emphasis placed on Markovian queues. Besides your class notes, supporting material for this homework is provided in Chapters 1 and 2 in your textbook, and also in Sections 8.1-8.6, 8.7.1 and 8.8.1 in the text by Cassandras and Lafortune. As I mentioned in class, both of these texts deal with a broader range of Markovian queueing stations than those that we had the time to address in detail in class, but the main lines of analysis and the underlying methodology are similar for all these stations. Also, your textbook provides (i) an exposition of some further computational tools and techniques that are available for these systems, and (ii) the derivation of some further results characterizing the steady-state operation of these environments. Some of these results are addressed in this homework.

B. Problems:

a. Solve Problems 8.4 and 8.5 in the text by Cassandras and Lafortune. (*Hint:* For Problem 8.5 refer to Section 8.6.6 of that textbook.)

b. Consider a taxi station where taxis and customers arrive in accordance with Poisson processes with respective rates of one and two per minute. A taxi will wait no matter how many other taxis are present. However, an arriving customer who does not find a taxi waiting leaves. Answer the following questions:

- i. What is the average number of taxis waiting?
- ii. What is the proportion of arriving customers that get taxis?
- iii. Answer questions (i) and (ii) above for the case where an arriving customer will leave only if the number of waiting customers exceeds a certain number N > 0. For this case, also compute the average waiting time in queue for those customers who join the waiting line.

iv. Answer question (iii) above in the case that the rate of the taxi arrivals is increased to two taxis per minute.

Hint: Try to model the various cases discussed in the above problem by a CTMC that has similar structure to the CTMC that models the operation of an M/M/1 queue. Then, use the perspectives and the results from the analysis of this last CTMC that was presented in class.

c. Consider a stable M/M/1 queueing station with arrival rate λ and processing rate μ that is operated at steady state. Show that the time in system, S, for a customer that is served by this station, follows an exponential distribution with rate $\mu - \lambda$.

d. Read the material on time reversibility of CTMCs attached to this homework. Then do the following:

- 1. Provide a proof for Proposition 1.6 in that material.
- 2. Use the result of Proposition 1.6 to argue that, in their steady-state regime, *ergodic* birth-death processes are time-reversible.
- 3. Use the result of part #2 above in order to provide an alternative proof for *Burke's* theorem. Also, show that this theorem generalizes to the M/M/m queueing stations.
- 4. Consider a stable FCFS M/M/1 queueing station with arrival rate λ and service rate μ . Use the results of parts #2 and #3 above in order to show that, in steady state, the number of customers that are encountered in this station by an arriving customer that eventually spends t time units in it, is Poisson distributed with mean equal to λt .
- e. Consider a single-server station with an infinite capacity queue, where:
 - the arrival process is characterized by a stochastic sequence $\{Y_1, Y_2, \ldots\}$ with r.v. Y_k characterizing the time elapsed between the k-1 and the k-th arrival;
 - the service process is characterized by a stochastic sequence $\{Z_1, Z_2, \ldots\}$ with r.v. Z_k denoting the time required for the processing of the k-th customer;
 - the server is operated in a *non-idling* mode;

• the queueing discipline is *preemptive Last-In-First-Out* (preemptive LIFO), i.e., a new arrival interrupts the processing of any job in the server, which is resumed only when all later arrivals have been cleared.

Express the time in system, S_k , for the k-th arrival at this system, in terms of the stochastic sequences $\{Y_1, Y_2, \ldots\}$ and $\{Z_1, Z_2, \ldots\}$.

f. Consider a $D/M/\infty$ queue where jobs arrive with a deterministic pace of one job every τ time units, and they enter immediately for service at one of the system servers. Processing times are exponentially distributed with rate μ . Answer the following:

- i. Compute the expected number of jobs in service that are encountered by a new arrival, as $t \to \infty$.
- ii. Compute the average number of jobs in service, as $t \to \infty$.
- iii. Which of the two quantities computed in parts (i) and (ii) above is larger? Provide a formal proof for your answer (and also an intuitive interpretation of your finding, if possible).
- iv. Finally, compute also the expected number of jobs that are in service upon the departure of some job, as $t \to \infty$.

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whereas with probability $1 - p_y$, there is no transition, i.e., the process will remain in state y, the same as at τ_{k-1} . Therefore, the sojourn time of \tilde{Y} in state y is a random summation: $\sum_{i=1}^{N} T_i$, where T_i 's are i.i.d. exponential variables with mean $1/\eta$ (interevent times of the Poisson process that generates the $\{\tau_k\}$ sequence), and N, independent of the T_i , is a geometric random variable with success probability p_y . It is easy to verify that this random summation also follows an exponential distribution, with rate $\eta p_y = \lambda(y) + \mu(y)$. Hence, this sojourn time distribution is exactly the same as in the original birth-death process $\{Y(t)\}$. From (1.5), it is also clear that if a transition takes place in state y, it is an upward transition with probability $\frac{\lambda(y)}{\lambda(y)+\mu(y)}$, and a downward transition with probability $\frac{\mu(y)}{\lambda(y)+\mu(y)}$. These are also the same as in $\{Y(t)\}$. Hence, starting from the same initial state, the two Markov chains $\{Y(t)\}$ and $\{\tilde{Y}(t)\}$ must have the same probability law.

1.2 Time Reversibility

Here we assume that the time index t belongs to the entire real line (instead of just the nonnegative half). Also, $\stackrel{d}{=}$ denotes equal in distribution.

Definition 1.1 A stochastic process $\{X(t)\}$ is *time-reversible* (or, *reversible*, for short) if

 $(X(t_1),\ldots,X(t_n)) \stackrel{\mathrm{d}}{=} (X(\tau-t_1),\ldots,X(\tau-t_n))$

for all t_1, \ldots, t_n , all n, and all τ .

Lemma 1.2 If $\{X(t)\}$ is reversible, then $\{X(t)\}$ is stationary. That is,

$$(X(t_1+\tau),\ldots,X(t_n+\tau)) \stackrel{\mathrm{d}}{=} (X(t_1),\ldots,X(t_n)),$$

for all t_1, \ldots, t_n , all n, and all τ .

Proof. Setting $\tau = 0$ in Definition 1.1, we have

$$(X(t_1),\ldots,X(t_n)) \stackrel{\mathrm{d}}{=} (X(-t_1),\ldots,X(-t_n)).$$

Next, replacing t_i by $t_i + \tau$ in Definition 1.1 for all i = 1, ..., n, we have

$$(X(t_1+\tau),\ldots,X(t_n+\tau)) \stackrel{\mathrm{d}}{=} (X(-t_1),\ldots,X(-t_n)).$$

Stationarity then follows from equating the left-hand sides of the above two equations. $\hfill \Box$

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1.2 Time Reversibility 5

Therefore, below we focus on stationary processes, in particular stationary Markov chains (or Markov chains in *equilibrium*, or in *steady state*).

Let $\{X(t)\}$ be a stationary Markov chain, with state space **S**. For ease of discussion, assume ergodicity, i.e., the (continuous-time) Markov chain is irreducible and positive recurrent. Recall that $\{X(t)\}$ is completely characterized by its rate matrix Q, whose entries are

$$Q_{ij} = q(i, j), \quad i \neq j,$$
$$-Q_{ii} = \sum_{j \neq i} q(i, j) := q(i),$$

where

$$q(i,j) = \lim_{h \to 0} \frac{1}{h} \mathsf{P}[X(h) = j | X(0) = i] := \lim_{h \to 0} P_h(i,j)/h,$$
(1.6)

and we assume $q(i) < \infty$ for all *i*. The stationary (or invariant) distribution, $\pi = (\pi(i))_{i \in S}$ is a vector of positive numbers (that sum to unity) satisfying

$$\pi(i) \sum_{j \neq i} q(i, j) := \pi(i)q(i) = \sum_{j \neq i} \pi(j)q(j, i),$$

or $\pi'Q = 0$ in matrix form. Note that under ergodicity, not only π is the limiting distribution of X(t) as $t \to \infty$, $\pi(i)$ is also the long-run average proportion of time that the Markov chain is in state *i*, for all *i*.

Now, even if $\{X(t)\}$ is not (necessarily) reversible, we can still define its time-reversal, $\{\tilde{X}(t)\}$, by letting $\tilde{X}(t) = X(\tau - t)$ for all t and for some τ . Since $\{X(t)\}$ is stationary, we can pick $\tau = 0$, for instance, without loss of generality. Note that while $\{X(t)\}$ evolves toward the right of the real line (the time axis), its time-reversal $\{\tilde{X}(t)\}$ evolves in the opposite direction, toward the left of the time line. It is easy to verify the Markov property of $\{\tilde{X}(t)\}$, as well as stationarity. It turns out that the Markov chain $\{X(t)\}$ and its time-reversal have some interesting relations.

Lemma 1.3 Let $\{X(t)\}$ be a stationary Markov chain with state space **S**, rate matrix Q, and stationary distribution π . Then, the time-reversal $\{\tilde{X}(t)\}$ is also a Markov chain, governed by a rate matrix \tilde{Q} , which is defined componentwise as follows:

$$\pi(i)\tilde{q}(i,j) = \pi(j)q(j,i), \quad \forall i,j \in \mathbf{S}, \ i \neq j;$$

$$(1.7)$$

Proof. First, observe that π is also the stationary distribution of $\{\tilde{X}(t)\}$, since

$$\mathsf{P}[X(t) = i] = \mathsf{P}[X(\tau - t) = i] = \pi(i)$$

ion, i.e., the process f_i the sojourn time of $e T_i$'s are i.i.d. expo f the Poisson process ent of the T_i , is a ge-. It is easy to verify tial distribution, with istribution is exactly t). From (1.5), it is f is an upward transiition with probability ice. starting from the $\{\tilde{Y}(t)\}$ must have the

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for all t, where the second equality follows from the stationarity of $\{X(t)\}$. For h > 0, letting $\tau = h$, we have

$$\pi(i)P_h(i,j) = \mathsf{P}[X(0) = i, X(h) = j] = \mathsf{P}[\tilde{X}(0) = j, \tilde{X}(h) = i] = \pi(j)\tilde{P}_h(j,i),$$

for any i, j. Dividing both sides by h and letting $h \to 0$ yields the desired relation in (1.7) [cf. (1.6)].

The above lemma implies that should $\tilde{Q} = Q$, then the time reversal \tilde{X} has the same probability law as the original process X; hence, X is *reversible*. In fact, the converse is also true (by mimicking the proof of the lemma); i.e., if X is reversible, then (1.7) holds with $\tilde{q} = q$.

Theorem 1.4 A stationary Markov chain $\{X(t)\}$ with state space **S** and rate matrix Q is reversible if and only if there exists a probability distribution on **S** satisfying

$$\pi(i)q(i,j) = \pi(j)q(j,i), \quad \forall i,j \in \mathbf{S}, \ i \neq j;$$

$$(1.8)$$

in which case π is the invariant distribution of $\{X(t)\}$.

Remark 1.5 The equations in (1.8) are called *detailed balance* equations, as opposed to the *full balance* equations that define the invariant distribution:

$$\pi(i)\sum_{j
eq i}q(i,j)=\sum_{j
eq i}\pi(j)q(j,i),\quad orall i\in {f S}.$$

(Note that the above are simply a row-by-row display of $\pi Q = 0$.) Obviously, detailed balance is stronger than full balance: Taking summation on both sides of (1.8) over $j \neq i$ yields the full balance equations.

Intuitively, full balance requires that the probability flow coming out of any given state, say i, to all other states—the *outflow*—be equal to the probability flow from all those other states going into the same state i—the *inflow*. In contrast, detailed balance insists that this balance be achieved at a more microscopic level: Outflow equals inflow between each pair of states $i \neq j$.

A (continuous-time) Markov chain (with rate matrix Q and state space **S**) is known to have a graphical representation: Let each state i be a node, and let the (directed) edge from i to j represent the transition rate q(i, j) if it is positive. If the transition rate satisfies

$$q(i,j) > 0 \Rightarrow q(j,i) > 0, \quad \forall i,j \in \mathbf{S},$$
(1.9)

then there is an (undirected) edge between i and j if and only if q(i, j) > 0.

Proposition 1.6 Suppose the stationary Markov chain $\{X(t)\}$ has transition rates that satisfy (1.9). Then, it is reversible if the associated graph is a tree.

Proof. Pick any pair ing (1.9), and the de q(i, j) > 0, then there the only probability words, the full balance detailed balance equa

Example 1.7 A spec associated graph is a u(i), respectively for jand the familiar relat

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Corollary 1.9 Support state space and is irrect rate matrix Q can be matrix and D a diagon

Example 1.10 Consider the constant rate λ and states and states process: X(t) den the constant rate $\{X(t)\}$ is a special case in equilibrium $\{X(t)\}$ is

Now consider the tim death queue, whose de inal queue, i.e., a Poiss reversibility, the origina abilistic behavior; and queues are exactly the process from a stationa: . exactly the same as t

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