ISyE7201: Production and Service System Engineering
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Spring 2016

Homework #3

Due Date: February 29, 2016

A. Reading assignment: This homework focuses on the theory of Continuous Time Markov Chains and their applications. An introductory treatment of this material is provided in Chpt. 1 of your textbook, but most of the material presented in class was drawn from Ross' and Resnick's books on stochastic processes. Also, some developments (especially the part of uniformization of CTMCs) were based on the text by Cassandras and Lafortune; CTMCs are covered in Sections 7.3-7.5 of that book.

B. Problems:

a. Solve Problems 7.16 and 7.19 from the text by Cassandras and Lafortune.

b. Consider a *pure birth* process with a constant birth rate $\lambda > 0$, i.e., a continuous time MC $\{X(t),\ t \geq 0\}$, with a countable state space $\mathcal{X} = \{0, 1, 2, \ldots\}$, and with the transition rates q_{ij} such that $q_{i,i+1} = \lambda,\ \forall i$, and 0, otherwise. For this chain, characterize the elements of the transition matrix $P(t)$, where $p_{ij}(t) \equiv \text{Prob}[X(t) = j | X(0) = i]$.

c. Consider a CTMC $\{X(t),\ t \geq 0\}$, with finite state space $\mathcal{X} = \{0, 1, \ldots, N\}$ and with rates q_{ij} such that $q_{i,i-1} = i\mu,\ \forall i \in \{1, 2, \ldots, N\}$, and 0 otherwise; it is further assumed that $\mu > 0$. Also, let $p_i(t) \equiv P[X(t) = i | X(0) = N]$ and show that $p_i(t)$ has a *binomial* distribution with parameters N and $e^{-\mu t}$.

Remark: A continuous time Markov chain presenting the above structure in terms of its transition rates, is characterized as a *pure death* process.

d. Currently, among the N individuals of a population, K have a certain infection that spreads as follows: Contacts between two members of the population occur in accordance with a Poisson process having rate λ. When a contact occurs, it is equally likely to involve any of the possible pairs of individuals in the population. If a contact involves an infected and a non-infected individual, then, with probability p the non-infected individual becomes infected. Once infected, an individual remains infected throughout.

Let $X(t)$ denote the number of infected members of the population at time t. Considering the current time as $t = 0$, show that the process $\{X(t), t \geq 0\}$ is a continuous-time Markov chain. Then, use this result in order to compute the expected time until all members of the considered population are infected.

e. Consider a sampling process where the obtained samples belong in m different categories. Each time a sample is taken, it belongs to category $i \in \{1, \ldots, m\}$ with probability p_i, and this outcome is independent from the outcomes of the previous sampling stages. Obviously, $\sum_{i=1}^{m} p_i = 1$.

Let N denote the number of samples that must be taken in order to obtain at least one sample from each category. Find $E[N]$.

f. At your local subway station, trains pass by every 5 minutes. Customers arrive at this station according to a Poisson process with rate $\lambda = 4$ customers per minute. What is the expected number of customers that you encounter when you arrive at the station?