ISyE7201: Production and Service System Engineering
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Spring 2016

Homework #2

Due Date: Wednesday, February 17, 2016

A. Reading assignment: This homework concerns the Poisson (counting) process and its properties. This material is covered in Sections 6.6 and 6.7 of Cassandras and Lafortune. But the class developments have also drawn from some other sources, primarily Ross' books on *"Stochastic Processes"* and *"Probability Models"*.

B. Problems:
a. Solve Problems 6.5, 6.6 and 6.11 from the text by Cassandras and Lafortune.

b. Let X be an exponential random variable. Without any computations, tell which one of the following is correct. Explain your answer.

1. $E[X^2|X > 1] = E[(X + 1)^2]$;
2. $E[X^2|X > 1] = E[X^2] + 1$;
3. $E[X^2|X > 1] = (1 + E[X])^2$.

c. One hundred items are simultaneously put on a life test. Suppose that the life-times of the individual items are independent exponential random variables with mean 200 hrs. The test will end when there have been a total of 5 failures. If T is the time when the test ends, find $E[T]$ and $Var[T]$.

d. Consider a two-service system in which a customer is served first by server 1, then by server 2, and then departs. The service times at server i are exponential random variables with rates $\mu_i,\ i = 1, 2$. When you arrive, you find server 1 free and two customers at server 2 – customer A in service and customer B waiting in the line.

1. Find P_A, the probability that customer A is still in service when you move over to server 2.

2. Find P_B, the probability that B is still in the system, when you move over to server 2.

3. Find $E[T]$, where T is the time that you spent in the system.

e. Two individuals, A and B, both require kidney transplants. If she does not receive a new kidney, A will die after an exponential time with rate μ_A, and B after an exponential time with rate μ_B. New kidneys arrive in accordance with a Poisson process having rate λ. It has been decided that the new kidney will go to A (or to B, if B is alive and A is not at that time) and the next one to B (if still living).

1. What is the probability that A obtains a new kidney?

2. What is the probability that B obtains a new kidney?

f. Consider the random variable $S = T_1 + T_2 + \ldots + T_n$, where $T_i,\ i = 1, \ldots, n$, are independent, exponentially distributed random variables with rate λ.

Show that the pdf of r.v. S is given by

$$f(s) = \begin{cases} \frac{\lambda^n s^{n-1} e^{-\lambda s}}{(n-1)!}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and its cdf is given by

$$F(s) = \begin{cases} 1 - e^{-\lambda s} \sum_{j=0}^{n-1} \frac{(\lambda s)^j}{j!}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Also, the mean of this random variable is

$$E[S] = n/\lambda$$

its variance is

$$Var[S] = n/\lambda^2$$

and its squared coefficient of variation is

$$SCV[S] \equiv Var[S]/(E[S])^2 = 1/n$$

Remark: Random variable S is said to possess an n-*Erlang* distribution. The Erlang distribution can be used in order to approximate any other distribution with $SCV << 1$, while retaining the Markovian structure that enables the system analysis through CTMC theory. Given a r.v X with mean $E[X]$ and $SCV[X] << 1$, n can be picked so that $SCV[S] = 1/n \approx SCV[X]$, and subsequently λ is picked so that$E[S] = n/\lambda = E[X]$. This approximating scheme essentially tries to match only the first two moments of the original and the approximating random variables. More degrees of freedom, and therefore, better approximating capabilities are obtained by letting the distribution of each r.v. $T_i,\ i = 1, \ldots, n$, have a different rate λ_i. The resulting distribution is known as a *hypo-exponential* distribution, and it will be shown later in class that its SCV is still less than or equal to one (the latter in case that $n = 1$). Obviously, the n-stage hypo-exponential distribution subsumes the n-Erlang distribution.

g. Consider two *independent* exponentially distributed random variables X_1 and X_2 with corresponding rates μ_1 and μ_2. Compute $E[X_2 | X_2 > X_1]$.