ISyE7201: Production and Service System Engineering
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Spring 2016

Homework #1

Due Date: February 8, 2016

Reading Guidelines
The introductory material on stochastic processes was (primarily) extracted from Section 6.2 in the text on Discrete Event Systems by Cassandras and Lafortune. Also, Discrete-Time Markov Chains (DTMCs) are treated in the same text in Section 7.2. But some of the material presented in class was also based on personal notes. Of course, a large part of the discussion on DTMCs had a reviewing role of the material on DTMCs that you had covered in Stoch I. Some of the problems assigned below have a similar flavor.

You are also invited to read Sections 6.3–6.5 from the text by Cassandras and Lafortune, but this will not be considered as part of the covered material. Such reading will give you some further perspective on how stochastic processes are used in broader systems theory and control. On the other hand, it is also true that some of this material requires some familiarity with automata concepts that are addressed in earlier chapters of that book, and might not be in your current background.

Problem set

A. Solve problems 6.1, 7.3, 7.7, 7.11 and 7.12 from the text by Cassandras and Lafortune, taking into consideration the following remark:

Remark For Problem 6.1 the correct recursion is:

$$X_{k+1} = X_k + S_{k+1}, \quad S_0 = 0, \; X_0 = 0$$

B. Consider a *synchronous* production line with N stations. At any operational cycle, the line has a failing probability p_f. Each such failure is classified in one of K categories with corresponding probabilities q_k, $k = 1, \ldots, K$ (obviously $\sum_{k=1}^{K} q_k = 1.0$). These failing modes are non-destructive, i.e., when the line gets into any failing mode $k \in K$, it remains still for a sequence of operational cycles until it is repaired, at which point it resumes its advancement. A line failing in mode k has a probability of being repaired in the current operational cycle equal to r_k.

Model the operation of this line as a Discrete-Time Markov Chain (DTMC) and use this model to compute the long-term throughput of the line.

C. Consider a DT-MC with finite state space $S = \{1, 2, \ldots, N\}$. In class we have provided a characterization of the recurrence or transience of any state i of it, based on the value of the following probability:

$$\rho_i \equiv \sum_{k=1}^{\infty} P[T_{ii} = k] = P[T_{ii} < \infty]$$

where T_{ii} denotes the recurrence time for state i.

(i) Let $p_{ii}^{(n)}$ denote the (i,i)-element of the n-th power of the one-step transition probability matrix P of the chain, and show that a state $i \in S$ of this chain is recurrent if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$.

(ii) Use the result in part (i) in order to show that if states i and j communicate and i is recurrent, then j must be recurrent as well (this essentially establishes the fact that *recurrence* is a *class* property).

(iii) Use the result in part (ii) to show that *transience* is also a *class* property for this chain.

Remark: The above problem asks you to prove some of the structural results for DT-MCs that we mentioned in class. You should have seen the corresponding proofs in Stoch I, so, this problem is more of a revision of the corresponding material.

D. Consider a finite-space DT-MC with a single absorbing state s_0 (this is, for instance, a structure that arises in "stochastic shortest path" problems, when the routing policy is fixed). Letting $s_1, \ldots, s_n$ denote the remaining transient states of the chain, the one-step transition probability of this chain, P, can be expressed as:

$$P = \begin{bmatrix} 1 & \mathbf{0}^T \\ P_a & P_T \end{bmatrix} \tag{1}$$

In Equation 1: $\mathbf{0}$ is the n-dim(ensional) zero (column) vector and $\mathbf{0}^T$ is its transpose; P_T is an n-dim square matrix characterizing the one-step transition probabilities of the chain among its transient states; P_a is an n-dim vector characterizing the one-step absorption probabilities of the process from the corresponding transient states; and 1 is a scalar quantity.

Let $f_i,\ i = 1, \ldots, n$ denote the "expected absorption time" of the chain when starting from transient state i, i.e., $f_i = E[T_{i0}],\ i = 1, \ldots, n$, where

$T_{i0}, i = 1, \ldots, n$, are the corresponding "first passage" or "hitting" times defined in class. Also, let the (column) vector $\mathbf{f}$ collect all the f_i's.

(i) Show that

$$\mathbf{f} = (I - P_T)^{-1}\mathbf{1} \tag{2}$$

where I is the $n \times n$ identity matrix and $\mathbf{1}$ is the $n \times 1$ (column) vector with all its components equal to 1.

(ii) Can you provide an intuitive interpretation for the result of Equation 2, based on an interpretation of the matrix $(I - P_T)^{-1}$?

E. Investigate what is the behavior of a picking line that is organized according to the bucket brigade policy, if there are pairs of consecutive pickers with equal velocities (i.e., there exist some i's with $v_i = v_{i+1}$).

Hint: Consider how the above assumption impacts the structure of the recursive equation that was used in class for describing the dynamics of the bucket brigade policy; in particular investigate the implications of this assumption for the structure of the non-negative square matrix that appears in this equation. What is the significance of these structural changes for the invocation of the Perron-Frobenius theorem in this new case?