ISYE 7201: Production & Service Systems
Spring 2015
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Midterm Exam
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Name: SOLUTIONS

Problem 1 (20 points): In the "gambler's ruin" problem, a gambler plays a game of chance where at each round she wins 1 unit with probability $p \in (0,1)$ and loses one unit with probability $1-p=q$. The gambler starts with i units, and the game ends when the gambler reaches N units (where $N > i$) and wins the game, or she ends up with 0 units, in which case she looses the game. It can be shown that, for such a game, the winning probability is given by

$$f_{iN}(p) = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 0.5 \\ \frac{i}{N} & \text{if } p = 0.5 \end{cases}$$

Use the above result, in order to address the following problem: A token moves around a ring of five nodes, numbered from 0 to 4 and with the numbering taking place in the counter-clockwise sense. At any node i, the token's next move is a counter-clockwise move, to node $(i+1)$ mod 5, with probability $p = 0.5$ and a clockwise move, to node $(i-1)$ mod 5, with the remaining probability $q = 1-p = 0.5$. Currently, the token is at node 0. We want to compute the probability that in its subsequent moves, the token will visit node 3 for the first time only after it has visited nodes 1, 2 and 4 (and, of course, node 0, which is its current location; also all the other four nodes might be visited more than once before the ultimate visit to node 3).

Hint: Start by expressing the considered event E in terms of the "hitting times" T_1, T_2, T_3 and T_4 for the corresponding nodes, and then process this expression to see how you can express the occurrence probability of E in terms of the provided result for the gambler's ruin problem.

Let T_i, $i=1,2,3,4$, denote the hitting time of node i. Then, the considered event can be expressed as:

$$P(E) = P(T_3 > T_2 \wedge T_3 > T_4) = P(T_3 > \max\{T_2, T_4\}) =$$

$$= P(T_3 > \max\{T_2, T_4\} \mid T_2 < T_4)\, P(T_2 < T_4) +$$

$$+ P(T_3 > \max\{T_2, T_4\} \mid T_2 > T_4)\, P(T_2 > T_4) =$$

$$= P(T_3 > T_4 \mid T_2 < T_4)\, P(T_2 < T_4) +$$

$$+ P(T_3 > T_2 \mid T_2 > T_4)\, P(T_2 > T_4)$$

But it can be easily checked that for $p = q = 0.5$:

$$\left.\begin{aligned} P(T_2 < T_4) &= f_{13}(0.5) = \frac{1}{3} \\ P(T_2 > T_4) &= 1 - P(T_2 < T_4) = 2/3 \\ P(T_3 > T_4 \mid T_2 < T_4) &= f_{14}(0.5) = 1/4 \\ P(T_3 > T_2 \mid T_2 > T_4) &= f_{14}(0.5) = 1/4 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \quad P(E) = \frac{1}{4} \times \frac{1}{3} + \frac{1}{4} \times \frac{2}{3} = \frac{1}{4}$$

Problem 2 (30 points): Buses arrive at a certain stop according to a Poisson process with rate λ. If you take the bus from the stop, then it takes a time R, measured from the time at which you entered the bus, to arrive home. If you walk from the bus stop, then it takes a time W to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to a time s, and if a bus has not yet arrived by that time then you walk home.

i. (15 pts) Compute the expected time from when you arrive at the bus stop until you reach home, under the applied policy.

ii. (15 pts) What choices of the policy parameter s minimize the expected time that you computed in part (i)? Can you provide an intuitive explanation for your result?

Let the r.v D denote the considered time and the r.v. T denote the time till the next bus arrival. Then, D is a function of T, and

$$E_T[D(T)] = \int_0^\infty D(t)\lambda e^{-\lambda t}dt =$$

$$= \lambda\int_0^s (t+R)e^{-\lambda t}dt + \lambda(s+W)\int_s^\infty e^{-\lambda t}dt$$

$$= \lambda\int_0^s te^{-\lambda t}dt + \lambda R\int_0^s e^{-\lambda t}dt + \lambda(s+W)\int_s^\infty e^{-\lambda t}dt$$

$$= \lambda\int_0^s te^{-\lambda t}dt + R[1-e^{-\lambda s}] + (s+W)e^{-\lambda s}$$

$$= \lambda\int_0^s te^{-\lambda t}dt + R\cdot P(T\leq s) + (s+W)P(T>s) \quad (1)$$

It is interesting to consider the structure of (1).

* The third term in this expression is $E[D(T) \mid T>s]\cdot P(T>s)$
* The second term is $E[$Time to go home once you get on the bus $\mid$ You got on the bus$]\cdot P[$You got on the bus$]$
* Hence, the first term must be $E[$Time for bus arrival $\mid T\leq s]\,P(T\leq s)$

Indeed,

$$\lambda \int_0^s t\, e^{-\lambda t}\, dt = \left[\int_0^s t \,\frac{\lambda e^{-\lambda t}}{1-e^{-\lambda s}}\, dt\right]\left[1-e^{-\lambda s}\right] =$$

$$= \left(\int_0^s t\, f_{T\le s}(t)\, dt\right)\cdot P(T\le s) = E[T \mid T\le s]\cdot P(T\le s)$$

Also, through integration by parts, we get:

$$\lambda \int_0^s t e^{-\lambda t}\, dt = \int_0^s e^{-\lambda t}\, dt - \left[t e^{-\lambda t}\right]_0^s =$$

$$= \frac{1}{\lambda}\left[1-e^{-\lambda s}\right] - s e^{-\lambda s} \qquad (2)$$

From (1) and (2):

$$E_T[D(T)] = \frac{1}{\lambda} + R + e^{-\lambda s}\left[s + W - R - 1/\lambda - s\right] =$$

$$= \frac{1}{\lambda} + R + \left(W - \frac{1}{\lambda} - R\right) e^{-\lambda s} \qquad (3)$$

From (3) we see that $E_T[D(T)]$ is minimized by choosing s according to the following rule:

(a) If $W - \frac{1}{\lambda} - R > 0$, we should set the last term in Eq. (3) to 0 by setting $s \to \infty$.

(b) If $W - \frac{1}{\lambda} - R < 0$, we want to maximize the reducing impact of the third term by setting $e^{-\lambda s}$ to its maximal possible value. This value is 1 and is obtained by setting $s = 0$.

(c) If $W - 1/\lambda - R = 0$, then obviously $E_T[D(T)]$ does not depend on s.

The natural interpretation for the last set of results is that if the time W that we need to walk home is larger than the expected time waiting for the next bus plus the time it takes the bus to drive to our home, then we should just wait for the bus. In the opposite case, we should start walking immediately! .)

Finally, if the above two times are equal, we can do whatever we want!

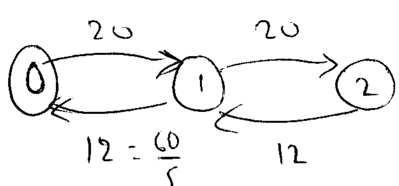
Problem 3 (30 points): Potential customers arrive at a full-service, single-pump gas station at a Poisson rate of 20 cars per hour. However, customers will only enter the station for gas if there are no other waiting cars at the pump (except possibly for the car currently being attended to). Suppose the amount of time required to service a car is exponentially distributed with a mean of five minutes.

i. (10 pts) Model the operation of this gas station as a continuous-time Markov chain.

ii. (10 pts) What fraction of the attendant's time is spent servicing cars?

iii. (10 pts) What fraction of arriving customers are lost?

(i) The operation at the considered gas station can be modeled by the following CTMC where the state is defined by the number of cars at the station



The corresponding infinitesimal generator is:

$$Q = \begin{bmatrix} -20 & 20 & 0 \\ 12 & -32 & 20 \\ 0 & 12 & -12 \end{bmatrix}$$

and the equations providing the steady-state probabilities are:

$$\begin{cases} -20 p_0 + 12 p_0 = 0 \\ 20 p_1 - 12 p_2 = 0 \\ p_0 + p_1 + p_2 = 1 \end{cases} \Rightarrow \begin{cases} p_0 = 9/49 \\ p_1 = 15/49 \\ p_2 = 25/49 \end{cases}$$

(ii) This is the fraction of time that the considered CTMC is at the states 1 and 2, i.e.,

$$P_1 + P_2 = \frac{15}{49} + \frac{25}{49} = \frac{40}{49}$$

(iii) Obviously the Poisson process characterizing the car arrivals is split into two subprocesses, that tracing the cars that enter the station and the complementary process that traces the cars leaving unserved. The classification of each arrival depends on whether this arrival finds the station in state 2. But in steady-state, the corresponding probability is $P_2 = \frac{25}{49}$

* Also notice that the two aforementioned sub-processes are also Poisson with rates $\lambda_1 = 20 \times \frac{24}{49}\ hr^{-1}$ and $\lambda_2 = 20 \times \frac{25}{49}\ hr^{-1}$.

Problem 4 (20 points): Consider an ergodic continuous-time Markov chain $\{X(t),\ t \geq 0\}$ defined over the state space $S = \{0, 1, 2, \ldots\}$. The instantaneous transitions rates for this process are denoted by $\{v_i,\ i = 0, 1, \ldots\}$ and the stationary probabilities are denoted by $\{P_i,\ i = 0, 1, \ldots\}$. Also, suppose that $X(0) = 0$, and let T denote the time point that, for the first time, the chain visits state 0 and stays there for t consecutive time units. Characterize the expectation $E[T|X(0) = 0]$ in terms of v_0 and P_0. Does your result extend to the case where the process $X(t)$ is a semi-Markov process?

Hint: Consider the characterization of the expected length $E[T_{jj}]$ of the recurrence cycles for any state j of an ergodic CTMC that we developed in class, during the proof of the results of the limiting regime of these processes. How many such cycles are needed, in expectation, to satisfy the condition that is posed by this problem?

Letting S_0 denote the random sojourn time at state $\emptyset$, we have $\text{Prob}(S_0 \geq t) = e^{-v_0 t} \equiv p$.

Hence, the expected # of visits to state $\emptyset$ until the realization of the desired events is equal to $1/p$

Furthermore, since we start from state $\emptyset$, the expected # of recurrence cycles back to this state till the occurrence of the desired event is $1/p - 1$.

The expected length of each of these cycles is

$$E[T_{00}] = \frac{\sum_\ell \pi_\ell \tau_\ell}{\pi_0} = \frac{\tau_0}{P_0} = \frac{1}{P_0 v_0}$$

Putting everything together, we get

Expected time till the successful visit =

$$= \left(\frac{1}{p} - 1\right) \frac{1}{P_0 v_0} = \left(e^{v_0 t} - 1\right) \frac{1}{P_0 v_0}$$

In the case where the considered process is semi-Markov, we still have:

Expected time till the successful visit =

$$\left(\frac{1}{P(S_0 \geq t)} - 1 \right) \frac{\tau_0}{P_0}$$

But $P(S_0 \geq t)$ and $\tau_0 = E[S_0]$ will have different forms that will depend on the distribution that characterizes the sojourn time at state 0.