ISYE 7201: Production & Service Systems
Spring 2015
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Final Exam
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Name: Solutions

Problem 1 (20 points): A doctor has scheduled two appointments, one at 1:00pm and the other at 1:30pm. The amounts of time that the appointments last, are independent exponential random variables, with mean 30 minutes. Assuming that both patients are on time, find the expected amount of time that the 1:30 appointment spends at the doctor's office.

Let T_i, $i=1,2$, the r.v. denoting the service time of patient i. Then,

Prob (Patient 2 finds Patient 1 in service upon arrival) $=$ Prob $(T_1 > 30 \text{ min}) =$
$e^{-30/30} = e^{-1}$

and

$$E[T_2] = (1-e^{-1})\,E[T] + e^{-1}(2E[T])$$
$$= E[T] + e^{-1}E[T] = (1+e^{-1})E[T]$$
$$= (1+e^{-1})\,30 \text{ min} = 41.036 \text{ min}$$

In the second computation we have used the fact that the memoryless property of the exp. distr. implies that the remaining service time of patient 1 if he is not done by the time that patient 2 arrives, is exp. distributed with the same rate as the rate of the exp. distr. characterizing the entire service time.

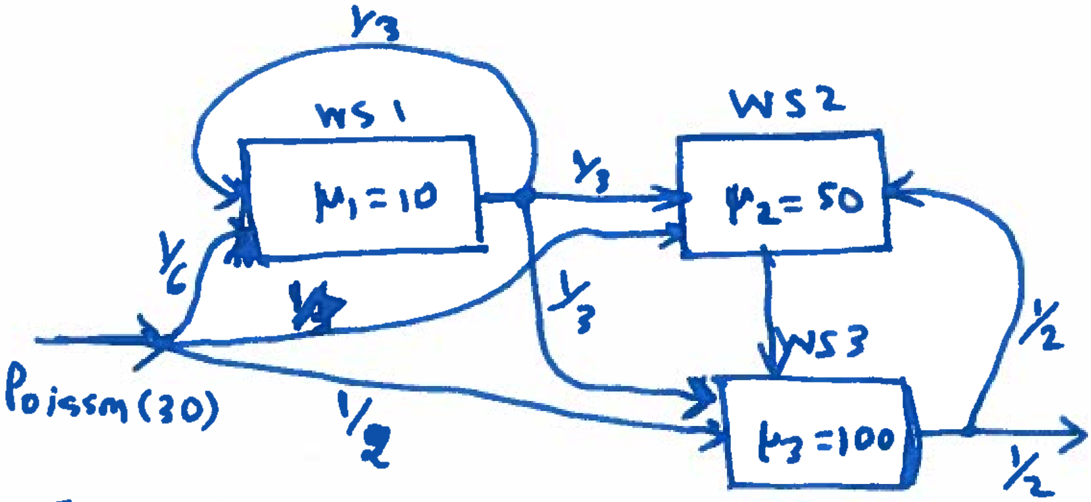
Problem 2 (30 points): Consider a network of three stations with a single server at each station. Customers arrive at stations 1, 2 and 3 in accordance with Poisson processes having respective rates 5, 10 and 15. The service times at the three stations are exponential with respective rates 10, 50 and 100. A customer completing service at station 1 is equally likely to go (i) go to station 2, (ii) go to station 3, or (iii) return to station 1. A customer departing service at station 2 always goes to station 3. A departure from station 3 is equally likely to either go to station 2 or leave the system.

i. (10 pts) What is the average number of customers in the system (i.e., in all three stations)?

ii. (10 pts) What is the average time a customer spends in this system?

iii. (10 pts) What is the expected number of visits to station 1 by any customer that enters this network?

The considered network:



The traffic equations:

$$\begin{cases} \lambda_1 = 5 + \frac{1}{3}\lambda_1 \\ \lambda_2 = 10 + \frac{1}{3}\lambda_1 + \frac{1}{2}\lambda_3 \\ \lambda_3 = 15 + \frac{1}{3}\lambda_1 + \lambda_2 \end{cases} \Rightarrow \dots \begin{cases} \lambda_1 = 15/2 = 7.5 \\ \lambda_2 = \frac{85}{2} = 42.5 \\ \lambda_3 = 60 \end{cases}$$

Then, the traffic intensities at each station are:

$$\rho_1 = \lambda_1/\mu_1 = \frac{7.5}{10} = 0.75$$

$$\rho_2 = \lambda_2/\mu_2 = \frac{42.5}{50} = 0.85$$

$$\rho_3 = \lambda_3/\mu_3 = \frac{60}{100} = 0.6$$

and from the independence of the three workstations at steady-state we get:

(i)
$$\left.\begin{aligned} L_1 &= \frac{\rho_1}{1-\rho_1} = \frac{0.75}{0.25} = 3 \\ L_2 &= \frac{\rho_2}{1-\rho_2} = \frac{0.85}{0.15} = 5.67 \\ L_3 &= \frac{\rho_3}{1-\rho_3} = \frac{0.6}{0.4} = 1.5 \end{aligned}\right\} \Rightarrow L = L_1 + L_2 + L_3 = 10.17$$

(ii) From Little's law applied to the entire network:

$$W = \frac{L}{TH} = \frac{10.17}{30} = 0.339$$

(iii) A customer will be visiting WS1 during its sojourn in this network only if it is routed to this workstation upon its entrance to the network. In that case, the customer will cycle into this workstation until ~~s/he~~ eventually leaves for the remaining part of the network. This last event happens with prob. $2/3$ at any visit. Hence, the expected # of visits given that the customer will visit WS1 is $3/2 = 1.5$.

And therefore, the expected # of visits to WS1 for any customer entering the network is

$\frac{1}{6} \times 1.5 = 0.25.$

Problem 3 (20 points): In a stable $M/G/1$ queue,

i. (10 pts) what proportion of departures leave the queue with zero workload?

ii. (10 pts) what is the average remaining workload seen by a departure?

Please, define carefully the notation that you use in your responses.

This problem ~~relies~~ relies heavily on the facts that in a stable $M/G/1$ queue

a) the steady-state distribution of the number of customers observed in the system by a departure is equal to the steady-state distr. of the number of customers in system observed by an arrival (because departures and arrivals for this queue occur one at a time);

b) The second distr. mentioned above is equal to the steady-state distr. of the number of customers in system (observed at a random time), due to PASTA).

Hence, the proportion requested in (i) is equal to the prob. that the system is empty at steady-state, i.e., $1-\rho$, where ρ is the traffic intensity of the qu

Similarly, the average remaining workload observed by a departure is the average workload in steady state, i.e.,

$$t \cdot \lambda \left(\frac{1+C_B^2}{2} \frac{\rho}{1-\rho} + 1 \right) t =$$

$$= \left(\frac{1+C_B^2}{2} \frac{\rho^2}{1-\rho} + \rho \right) t$$

Problem 4 (30 points): Consider an $M/G/1$ queue where parts arrive at a rate of λ parts per time unit, but the parts are processed in batches of k parts per batch. The expected batch processing time is equal to t time units, and the corresponding coefficient of variation is equal to c_B. Adapt the mean value analysis for the $M/G/1$ and $G/G/1$ queues presented in class to this new case. In particular, perform the following tasks:

i. (10 pts) First determine the stability condition for this new queue. What are the implications of this condition for the minimal batch size?

ii. Next assume a stable instantiation of the considered queue that operates in steady-state, and compute the following performance measures:

   (a) (10 pts) The expected cycle time for an arriving part.

   (b) (5 pts) The throughput of this queue.

   (c) (5 pts) The average number of parts in this queue.

Remark:— Notice that, in general, an arriving part goes through three phases during its sojourn time in this queue: (i) Unless it is the last part of a newly formed batch, it will wait for a certain time until a complete batch is formed; (ii) then, it will move, together with the rest of its batch, into a queue of batches waiting to be processed at the by the server; and (iii) it will go into the server to be processed with the rest of the parts in its batch. Compute the expected cycle time by computing the expected time length for each of these three phases.

(i) The arrival rate per batch is λ/k and the batch proc. time is t; hence the stability condition is: $\frac{\lambda}{k} t < 1 \Rightarrow \underline{\lambda t < k}$

It is interesting to notice that this condition dictates a <u>minimum</u> batch size.

(ii)

(a) following the breakdown of the cycle time that is provided in hint following the problem statement above, we have:

$$CT = WTTB + W_q + t$$

where

- WTTB denotes the expected time waiting by a part for the formation of its batch;
- W_q is the exp time waiting in the batch queue; and
- t is the batch proc. time.

A simple way to compute WTTB is as follows:

(I) the first arriving unit for a new batch will wait for $k-1$ other arrivals for a total exp. time of $\frac{1}{\lambda}(k-1)$

(II) the second arriving unit will experience an exp. waiting time till the batch formation of $\frac{1}{\lambda}(k-2)$

(III) the next-to-last unit will wait for $1/\lambda$ time units

(IV) and the last arrival will experience a waiting time of 0

Averaging the above delays, we have:

$$WTTB = \frac{1}{k}\frac{1}{\lambda}\left[(k-1)+(k-2)+\dots+1\right] =$$

$$= \frac{1}{k}\frac{1}{\lambda}\frac{(k-1)k}{2} = \frac{k-1}{2\lambda}$$

For W_q we can apply the corresponding Kingman's approximation for the G/G/1 queue; i.e.,

$$W_q = \frac{C_{a_b}^2 + C_B^2}{2} \frac{\rho}{1-\rho} t$$

where

- $\rho = \frac{\lambda}{k} t = \frac{\lambda t}{k}$

- $C_{a_b}^2 = \frac{\text{Variance of batch inter-arrival times}}{(\text{mean batch inter-arrival time})^2} =$

$$= \frac{k \, 1/\lambda^2}{(k \, 1/\lambda)^2} = \frac{1}{k}$$

Putting everything together we get:

$$CT = \frac{k-1}{2\lambda} + \frac{1+kC_B^2}{2} \frac{\lambda t}{k - \lambda t} \frac{t}{k} + t$$

(ii) For a stable such queue, the conservation of material implies that $TH = \lambda$.

(iii) From Little's law applied to the entire queue:

$$WIP = TH \cdot CT = \lambda \cdot CT$$