Example of closed QNs

- Two customers move among two single-server stations.
- After each ~~complete~~ service completion, the customer is equally likely to go to either station for the next service.
- Service times exp. distributed with rates μ_i, $i=1,2$.

(a) Determine the average # of customers at each station

(b) " " service completion rate (i.e, the throughput) at each station

(c) Compute the st-state prob. that station i, $i=1,2$, will have 2 customers in it.

Solution:

First, compute the (relative) ~~[illegible]~~ arrival rates at each station:

$$\begin{cases} v_1 = v_1 \tfrac{1}{2} + v_2 \tfrac{1}{2} \\ v_2 = v_1 \tfrac{1}{2} + v_2 \tfrac{1}{2} \end{cases} \rightarrow v_1 = v_2$$

Pick $v_1 = v_2 = 1$.

Next, we answer (a) and (b) through MVA.

- For $N=1$:

$$W_i(1) = 1/\mu_i \quad (\text{from Eq. } 3)$$

Also $TH(1) = \dfrac{1}{\sum_{j=1}^{2} v_j W_j(1)}$ this is the throughput function

Hence, $$L_i(1) = \frac{1 \cdot v_i\, W_i(1)}{\sum_{j=1}^{2} v_j W_j(1)} = \frac{1/\mu_i}{1/\mu_i + 1/\mu_j} = \frac{\mu_j}{\mu_i + \mu_j}$$ (the slowest server has more customers in expectation)

or (from the above and Little's law):

$$TH_i(1) = L_i(1) / W_i(1) = \frac{\mu_j/(\mu_i+\mu_j)}{1/\mu_i} = \frac{\mu_i \mu_j}{\mu_i + \mu_j} = \frac{1}{1/\mu_i + 1/\mu_j}$$

②

for $N=2$

$$- W_i(2) = \frac{1}{\mu_i}\left[l_i(1)+1 \right] = \frac{1}{\mu_i}\left[\frac{\mu_j}{\mu_i+\mu_j} + 1 \right] = \frac{2\mu_j+\mu_i}{\mu_i(\mu_i+\mu_j)}$$

$$- l_i(2) = \frac{2 v_i W_i(2)}{\sum_{j=1}^{2} v_i W_j(2)} = \frac{2\,\frac{2\mu_j+\mu_i}{\mu_i(\mu_i+\mu_j)}}{\frac{2\mu_j+\mu_i}{\mu_i(\mu_i+\mu_j)} + \frac{2\mu_i+\mu_j}{\mu_j(\mu_i+\mu_j)}} =$$

$$= \frac{2(2\mu_j+\mu_i)\cdot\mu_j}{(2\mu_j+\mu_i)\mu_j + (2\mu_i+\mu_j)\mu_i} = \frac{2(2\mu_j+\mu_i)\mu_j}{2(\mu_j^2+\mu_i^2+\mu_i\mu_j)} = \frac{(2\mu_j+\mu_i)\mu_j}{\mu_j^2+\mu_i^2+\mu_i\mu_j}$$

$$- TH_i(2) = l_i(2)/W_i(2) = \frac{(2\mu_j+\mu_i)\mu_j/(\mu_j^2+\mu_i^2+\mu_i\mu_j)}{(2\mu_j+\mu_i)/\mu_i(\mu_i+\mu_j)} =$$

$$= \frac{\mu_i\mu_j(\mu_i+\mu_j)}{\mu_j^2+\mu_i^2+\mu_i\mu_j}$$

Special case: $\mu_i=\mu_j=\mu$

$$- W_i(2) = \frac{3\mu}{\mu(2\mu)} = \frac{3}{2}\frac{1}{\mu} \quad \text{(from the arrival thm)} \qquad\qquad - W_i(1) = 1/\mu$$

$$- l_i(2) = \frac{3\mu^2}{3\mu^2} = 1 \quad (\checkmark \text{ from symmetry}) \qquad\qquad - l_i(1) = \frac{\mu}{2\mu} = 1/2$$

$$- TH_i(2) = \frac{2\mu^3}{3\mu^2} = \frac{2}{3}\mu \qquad\qquad - TH_i(1) = \frac{\mu^2}{2\mu} = 1/2\,\mu$$

Part c — 1st approach:

Prob (Station i has 2 customers) = Prob (Station j has $\emptyset$ customers)

= Prob (Server of station j is idle) = $1 - U_j(2)$

But $U_j(2) = TH_j(2)/\mu_j$

Therefore:

Prob (Station i has 2 customers) = $1 - TH_j(2)/\mu_j$ =

$$= 1 - \frac{\mu_i \mu_j (\mu_i + \mu_j)}{\mu_j(\mu_j^2 + \mu_i^2 + \mu_i \mu_j)} = \frac{\mu_i^2 + \mu_j^2 + \mu_i\mu_j - \mu_i^2 - \mu_i\mu_j}{\mu_i^2 + \mu_j^2 + \mu_i\mu_j} =$$

$$= \frac{\mu_j^2}{\mu_i^2 + \mu_j^2 + \mu_i\mu_j}$$

Special case: $\mu_i = \mu_j = \mu$

Prob(Station i has 2 customers) = $\dfrac{\mu^2}{3\mu^2} = \dfrac{1}{3}$

Part c – 2nd approach

$$P[X_i = n] = \left(\frac{v_i}{\mu_i}\right)^n \frac{G_{-i}(N-n)}{G(N)} \rightsquigarrow P[X_i = 2] = \left(\frac{1}{\mu_i}\right)^2 \frac{G_{-i}(0)}{G(2)}$$

Computing $G(2)$:

- $g(1,n) = \left(1/\mu_1\right)^n$, $n = 0,1,2$
- $g(j,0) = 1$, $j = 1,2$

from the simplified recursion for Buzen's algorithm: $g(j,n) = \rho_j g(j,n-1) + g(j-1,n)$

$$g(2,2) = \rho_2\, g(2,1) + g(1,2) =$$

$$= \rho_2 [\rho_2\, g(2,0) + g(1,1)] + g(1,2) =$$

$$= \rho_2^2\, g(2,0) + \rho_2\, g(1,1) + g(1,2) =$$

$$= \left(\frac{1}{\mu_2}\right)^2 + \left(\frac{1}{\mu_2}\right)\cdot\left(\frac{1}{\mu_1}\right) + \left(\frac{1}{\mu_1}\right)^2 = \cancel{\frac{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}{\mu_1\mu_2}}$$

Also

$$\cancel{g_{-i}(0) = 1 :}$$

Hence,

$$\cancel{P[X_i = 2] = \left(\frac{1}{\mu_i}\right)^2 \frac{\mu_i\mu_j}{\mu_i^2 + \mu_i\mu_j + \mu_j^2} = \frac{\mu_j}{\mu_i(\mu_i^2 + \mu_i\mu_j + \mu_j^2)}}$$

~~Special case:~~ $\mu_i = \mu_j = \mu$

$$P[X_i = 2] = \frac{1}{3}\cdot\frac{1}{\mu^2}$$

$$= \frac{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}{\mu_2^2 \mu_1^2}$$

Also: $G_{-i}(0) = 1$

Hence,

$$P[X_i = 2] = \left(\frac{1}{\mu_i}\right)^2 \frac{\mu_i^2 \mu_j^2}{\mu_i^2 + \mu_i\mu_j + \mu_j^2} = \frac{\mu_j^2}{\mu_i^2 + \mu_i\mu_j + \mu_j^2}$$