ISYE 6650: Probabilistic Models
Fall 2007
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Midterm Exam
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Name: SOLUTIONS

Problem 1 (20%) There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the three coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

Let us denote the various events of interest as follows:

2H = the flipped coin was two-headed
F = " " " " fair
B = " " " " biased to come up heads 75% of the time
H = the coin comes up heads

Then

$$P(2H|H) = \frac{P(2H, H)}{P(H)} =$$

$$= \frac{P(H|2H) \cdot P(2H)}{P(H|2H) \cdot P(2H) + P(H|F) \cdot P(F) + P(H|B) P(B)} =$$

$$= \frac{1 \cdot 1/3}{1 \cdot 1/3 + 1/2 \cdot 1/3 + 3/4 \cdot 1/3} = \frac{4}{9}$$

Problem 2 (20%) The number of customers entering a store on a given day is Poisson distributed with mean $\lambda = 10$. The amount of money spent by a customer is uniformly distributed over (0,100). Find the mean and variance of the amount of money that the store takes in on a given day.

Let

N = the number of customers entering the store during a day

X_i = the money spent by the i-th customer

S = the money collected by the store on that day

Then

$$S = \sum_{i=1}^{N} X_i$$

and it is a compound Poisson r.v. (since N is Poisson-distributed)

Hence, according to the results presented in class,

$$E[S] = E[N] \cdot E[X] = \lambda E[X] = 10 \cdot 50 = 500$$

$$Var[S] = \lambda \cdot E[X^2]$$

$$E[X^2] = \frac{1}{100} \int_0^{100} x^2 \, dx = \frac{100^3}{300} = \frac{10^4}{3}$$

and

$$Var[S] = \frac{10^5}{3}$$

Problem 3 (20%) In a production lot of 100 units there are 8 defective ones. We take a sample of 5 units from this lot.

i. What is the probability that there are exactly 3 defective parts in this sample?

ii. What is the probability that there are exactly 3 non-defective parts in this sample?

iii. What is the probability that there are at least 3 non-defective parts in this sample?

(i) $$\frac{\binom{92}{2}\cdot\binom{8}{3}}{\binom{100}{5}}$$

(ii) $$\frac{\binom{92}{3}\binom{8}{2}}{\binom{100}{5}}$$

(iii) $$\frac{\binom{92}{3}\binom{8}{2}+\binom{92}{4}\binom{8}{1}+\binom{92}{5}\binom{8}{0}}{\binom{100}{5}}$$

Remark: Notice that for case (i)

$$\frac{\binom{92}{2}\binom{8}{3}}{\binom{100}{5}} = \frac{\frac{92!}{2!\,90!}\;\frac{8!}{3!\,5!}}{\frac{100!}{5!\,95!}} = \frac{5!}{2!\,3!}\;\frac{91\cdot 92\cdot 6\cdot 7\cdot 8}{96\cdot 97\cdot 98\cdot 99\cdot 100} =$$

$$= \binom{5}{3}\frac{91\cdot 92\cdot 6\cdot 7\cdot 8}{100\cdot 99\cdot 98\cdot 97\cdot 96}$$

This last expression can be interpreted as:

$$\frac{8}{100}\cdot\frac{7}{99}\cdot\frac{6}{98}\cdot\frac{92}{97}\cdot\frac{91}{96}+\frac{8}{100}\cdot\frac{7}{99}\cdot\frac{92}{98}\cdot\frac{6}{97}\cdot\frac{91}{96}+\ldots$$

$\binom{5}{3}=10$ terms in total, each equal to $\frac{91\cdot 92\cdot 6\cdot 7\cdot 8}{100\cdot 99\cdot 98\cdot 97\cdot 96}$. Each of these terms expresses the probability of getting the defects in certain positions of the sampling sequence.

Problem 4 (20%)

i. Show that the *moment generating function* of a random variable X modeling a *Bernoulli trial* with success probability p is equal to $\phi_1(t) = pe^t + 1 - p$.

ii. Use the result in part (i) above, in order to show that the moment generating function of a *Binomial* distribution with parameters n and p is equal to $\phi_2(t) = (pe^t + 1 - p)^n$.

iii. Use the result of part (ii) above, in order to show that if X and Y are independent r.v.'s following Binomial distributions with respective parameters (n_X, p) and (n_Y, p), then the r.v. $X+Y$ follows a Binomial distribution with parameters $(n_X + n_Y, p)$.

iv. What is the intuitive interpretation of the result established in part (iii) above?

(i) $\varphi_X(t) = E[e^{tX}] = pe^t + (1-p)e^0 =$

$= pe^t + 1 - p$

(ii) Let $Y \sim$ Binomial(n, p) Then

$Y = \sum_{i=1}^{n} X_i$ where X_i, $i = 1, \dots, n$, are iid r.v.'s

distributed according to a Bernoulli distribution with success probability equal to p.

Hence $\varphi_Y(t) = (pe^t + 1 - p)^n$

(iii) $\varphi_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}]E[e^{tY}] =$

$= (pe^t + 1-p)^{n_X} (pe^t + 1 - p)^{n_Y} = (pe^t + 1 - p)^{n_X + n_Y} \Rightarrow$

$\Rightarrow X + Y \sim$ Binomial$(n_X + n_Y, p)$

(iv) $X + Y$ essentially models the number of successes in $(n_X + n_Y)$ Bernoulli trials with success prob. equal to p.

Problem 5 (20%) Let X be exponential with mean $1/\lambda$; that is

$$f_X(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty$$

i. Argue that

$$f_{X|X>1}(x) = \frac{\lambda e^{-\lambda x}}{P\{X > 1\}}, \quad 1 < x < \infty$$

ii. Use the result of part (i) to show that $E[X|X > 1] = 1 + \frac{1}{\lambda}$

iii. Can you provide an intuitive interpretation for the result of part (ii)?

(i) As discussed in class

$$f_{X|X>1}(x)\,dx \simeq \text{Prob}\{x \le X \le x + dx \mid X > 1\} =$$

$$= \frac{\text{Prob}\{x \le X \le x + dx,\ X > 1\}}{\text{Prob}\{X > 1\}} =$$

$$\begin{cases} \dfrac{\text{Prob}\{x \le X \le x + dx\}}{\text{Prob}\{X > 1\}}, & x > 1 \\ 0, & x < 1 \end{cases} =$$

$$= \begin{cases} \dfrac{f_X(x)}{\text{Prob}\{X > 1\}}, & x > 1 \\ 0, & x < 1 \end{cases}$$

A more formal argument for part (i) is as follows:

$$F_{X|X>1}(x) = P\{X \le x \mid X>1\} =$$

$$= \begin{cases} \dfrac{P\{1<X\le x\}}{P\{X>1\}}, & x>1 \\ 0, & \text{otherwise} \end{cases}$$

But

$$\frac{P\{1<X\le x\}}{P\{X>1\}} = \frac{P\{X>1\} - P\{X>x\}}{P\{X>1\}} =$$

$$= 1 - \frac{e^{-\lambda x}}{P\{X>1\}}$$

Finally

$$f_{X|X>1}(x) = \frac{dF_{X|X>1}(x)}{dx} = \begin{cases} \dfrac{\lambda e^{-\lambda x}}{P\{X>1\}}, & x>1 \\ 0, & \text{otherwise} \end{cases}$$

(ii) $$P\{X>1\} = \int_1^{\infty} \lambda e^{-\lambda x} dx = \lambda \int_1^{\infty} e^{-\lambda x} dx =$$

$$= -e^{-\lambda x}\Big|_1^{\infty} = -(0 - e^{-\lambda}) = e^{-\lambda}$$

Then

$$E[X \mid X>1] = \int_1^{\infty} x \lambda e^{-\lambda x} e^{\lambda} dx =$$

$$= \lambda e^{\lambda} \int_1^{\infty} x e^{-\lambda x} dx =$$

$$= \lambda e^{\lambda} \left[-x \frac{1}{\lambda} e^{-\lambda x} \right]_1^{\infty} + e^{\lambda} \int_1^{\infty} e^{-\lambda x} dx =$$

$$= \lambda e^{\lambda} \left[\frac{e^{-\lambda}}{\lambda} \right] - \frac{e^{\lambda}}{\lambda} \left[e^{-\lambda x} \right]_1^{\infty} =$$

$$= 1 + 1/\lambda$$

(iii) The result of part (ii) essentially implies that the expected value of the conditional distribution is equal to the expected value of the original (unconditional) distribution plus the lower bound for the value of X established by the conditioning. In other words,

$$E[X-1 \mid X>1] = 1/\lambda = E[X]$$

The above equation is a manifestation of the "memoryless" property of the exponential distribution.