Limiting Probabilities

Theorem: For an irreducible ergodic Markov Chain,

$\lim_{n\to\infty} P_{ij}^n$ exists and is independent of i.

Furthermore, letting

$$\pi_j = \lim_{n\to\infty} P_{ij}^n \quad , \quad j \geq 0 \qquad (1)$$

then, π_j is the unique, nonnegative solution of

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad j \geq 0 \qquad (\underline{\pi}^T = \pi^T P) \qquad (2)$$

$$\sum_{j=0}^{\infty} \pi_j = 1$$

Sketch of the Proof.

a) Consider the vector $u = (1\ 1\ \dots\ 1)^T$. For a stochastic matrix P, u is an eigenvector of P with eigenvalue equal to $\lambda = 1$.

Indeed $\forall i$: $P_{i\cdot} \cdot u = \sum_{j=1}^{\infty} P_{ij} = 1$

b) For stochastic matrices $\lambda = 1$ is the maximal eigenvalue, in terms of magnitude (Perron–Frobenius theorem)

c) For irreducible ergodic Markov Chains, the equation $Pu = u$ has a single solution upto a scaling factor, i.e, λ has a single multiplicity

d) Hence, P can be diagonalised as

$$P = [\,\underline{1}\ v_2\ \dots\ v_n\,] \begin{bmatrix} 1 & & \\ & \lambda_2 & \\ & & \ddots \ \lambda_n \end{bmatrix} [u_1\, u_2 \dots u_n]$$

where $\lambda_2 \dots \lambda_n$ are the remaining eigenvalues of P with $|\lambda| < 1$, $v_2, \dots, v_n$ are the corresponding eigenvectors, and

$[u_1\, u_2 \dots u_n] = [1\ v_2 \dots v_n]^{-1}$.

e) But then

$$P^n = [\underline{1}\ v_2 \ldots v_n] \begin{bmatrix} 1 & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_n^n \end{bmatrix} [u_1\ u_2 \ldots u_n]$$

and

$$\lim_{n\to\infty} P^n = [\underline{1}\ v_2 \ldots v_n] \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix} [u_1\ u_2 \ldots u_n] =$$

$$= \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \ldots & 0 \end{bmatrix} [u_1\ u_2 \ldots u_n] =$$

$$= \begin{bmatrix} u_{11} & u_{21} & \ldots & u_{n1} \\ u_{11} & u_{21} & \ldots & u_{n1} \\ - & - & & - \\ u_{11} & u_{21} & \ldots & u_{n1} \end{bmatrix}$$

f) To see that $u_{11} + u_{21} + \ldots + u_{n1} = 1$, simply notice that

$$[u_1\ u_2 \ldots u_n][\underline{1}\ v_1 \ldots v_n] = I \quad \text{(the } n\times n \text{ identity matrix)}$$

g) Finally, to prove (2), simply notice that

$$a_j^n = \sum_{i=1}^{\infty} a_i^{n-1} \cdot P_{ij}$$

and therefore

$$\lim_{n\to\infty} a_j^n = \sum_{i=1}^{\infty} \lim_{n\to\infty} a_i^{n-1} P_{ij} \Rightarrow$$

$$\Leftrightarrow \quad \pi_j = \sum_{i=1}^{\infty} \pi_i P_{ij}$$

Perron–Frobenius theorem

From Wikipedia, the free encyclopedia

In mathematics, the **Perron–Frobenius theorem**, named after Oskar Perron and Ferdinand Georg Frobenius, is a theorem in matrix theory about the eigenvalues and eigenvectors of a real positive $n \times n$ matrix:

Let $A = (a_{ij})$ be a real $n \times n$ matrix with positive entries $a_{ij} > 0$. Then the following statements hold:

1. there is a positive real eigenvalue r of A such that any other eigenvalue λ satisfies $|\lambda| < r$.
2. the eigenvalue r is simple: r is a simple root of the characteristic polynomial of A. In particular both the right and left eigenspace associated to r are 1-dimensional.
3. there is a left (respectively right) eigenvector associated with r having positive entries. This means that there exists a row-vector $v = (v_1, \ldots, v_n)$ and a column-vector $w = (w_1, \ldots, w_n)^t$ with positive entries $v_i > 0,\ w_i > 0$ such that $vA = rv, \quad Aw = rw$. The vector v (resp. w) is then called a left (resp. right) eigenvector associated with r. In particular there exist two uniquely determined left (resp. right) positive eigenvectors associated with r (sometimes also called "stochastic" eigenvectors) v_{norm} and w_{norm} such that $\sum_i v_i = \sum_i w_i = 1$.
4. one has the eigenvalue estimate $\min_i \sum_j a_{ij} \leq r \leq \max_i \sum_j a_{ij}$

The first statement says that the spectral radius of the matrix A coincides with r. The theorem applies in particular to a positive stochastic matrix. A right (respectively left) stochastic matrix A is a non-negative real matrix such that its row sums (respectively column sums) are all equal to 1. In this case the Perron–Frobenius theorem asserts that (provided all entries are strictly positive) the eigenvalue $\lambda = 1$ is simple and all other eigenvalues $\lambda \neq 1$ of A satisfy $|\lambda| < 1$. Also, in this case there exists a vector having positive entries, summing to 1, which is a right (resp. left) positive eigenvector associated to the eigenvalue $\lambda = 1$. Both properties can then be used in combination to show that the limit $A_\infty := \lim_{k \to \infty} A^k$ exists and is a positive stochastic matrix of matrix rank one. If A is left (resp. right) stochastic then A_∞ is again left (resp. right) stochastic. Its entries are determined by the stochastic left resp. right eigenvectors v_{norm} and w_{norm} introduced above. If A is right (resp. left) stochastic then the entry a_{ij} of A_∞ is equal to the jth entry of v_{norm} (resp. the ith entry of w_{norm}).

This result has a natural interpretation in the theory of finite Markov chains (where it is the matrix-theoretic equivalent of the convergence of a finite Markov chain, formulated in terms of the transition matrix of the chain; see, for example, the article on the subshift of finite type). More generally, it is frequently applied in the theory of transfer operators, where it is commonly known as the **Ruelle-Perron–Frobenius theorem** (named after David Ruelle). In this case, the leading eigenvalue corresponds to the thermodynamic equilibrium of a dynamical system, and the lesser eigenvalues to the decay modes of a system that is not in equilibrium.

The Perron–Frobenius theorem can be further generalized to the class of block-indecomposable non-negative matrices (called "irreducible" in reference [1] below, also called regular in the stochastic case). In particular it also holds if some positive power $B = A^k$, $k > 0$ of the non-negative matrix A has positive entries.

This generalization of the Perron-Frobenius theorem has particular use in algebraic graph theory. The "underlying graph" of a nonnegative real $n \times n$ matrix is the graph with vertices $1, \ldots, n$ and arc ij if and only if $A_{ij} \neq 0$. If the underlying graph of such a matrix is strongly connected, then the matrix is irreducible, and thus the generalized Perron-Frobenius theorem applies. In particular, the adjacency matrix of a connected graph is irreducible.

Perron-Frobenius theorem for non-negative matrices

Let $A = a_{ij}$ be a real $n \times n$ matrix with non-negative entries $a_{ij} \geq 0$. Then the following statements hold:

1. there is a real eigenvalue r of A such that any other eigenvalue λ satisfies $|\lambda| \leq r$. This property may also be stated more concisely by saying that the spectral radius of A is an eigenvalue.
2. there is a left (respectively right) eigenvector associated with r having non-negative entries.
3. one has the eigenvalue estimate $\min_i \sum_j a_{ij} \leq r \leq \max_i \sum_j a_{ij}$

With respect to the theorem above related to positive matrices, the left and right eigenvectors associated with the Perron root r are no longer guaranteed to be positive; but remain non-negative. Furthermore, the Perron root is no longer necessarily simple. If one requires the matrix A to be irreducible (its associated graph is connected) as well as non-negative, the eigenvector has (strictly) positive entries. Note that a positive matrix is irreducible (as its associated graph is fully connected) but the converse is not necessarily true. And if A is primitive ($A^k > 0$ for some k), then all the results above given for the case of a positive matrix apply.

References

1. R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1990 (chapter 8).
2. A. Graham, *Nonnegative Matrices and Applicable Topics in Linear Algebra*, John Wiley&Sons, New York, 1987.
3. Henryk Minc, *Nonnegative matrices*, John Wiley&Sons, New York, 1988, ISBN 0-471-83966-3
4. C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, 2001 (chapter 8).
5. A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, 1979 (chapter 2), ISBN 0-12-092250-9