Homework 3 Solution

1. **(Ross 3.5)**
(a) $P\{X = i|Y = 3\} = P\{i$ white balls selected when choosing 3 balls from 3 white and 6 red $\}$

$$= \frac{\begin{bmatrix} 3 \\ i \end{bmatrix} \begin{bmatrix} 6 \\ 3-i \end{bmatrix}}{\begin{bmatrix} 9 \\ 3 \end{bmatrix}}, i = 0, 1, 2, 3.$$

(b) By same reasoning as in (a), if $Y = 1$, then X has the same distribution as the number of white balls chosen when 5 balls are chosen from 3 white and 6 red. Hence,

$$E[X|Y] = 5\frac{3}{9} = \frac{5}{3}.$$

For a justification and a better understanding of the formula underlying this last result, refer to Examples 2.34 and 3.2 in your textbook.

2. **(Ross 3.7)**
Given $Y = 2$, the conditional distribution of X and Z is

$$P\{(X, Z) = (1, 1)|Y = 2\} = P\{X, Y, Z) = (1, 2, 1)\}/P\{Y = 2\} = 1/16(1/16+1/4) = \frac{1}{5}$$

In a similar manner, we get:

$$P\{(1, 2)|Y = 2\} = 0$$

$$P\{(2, 1)|Y = 2\} = 0$$

$$P\{(2, 2)|Y = 2\} = \frac{4}{5}.$$

Notice that, as expected, $\sum_x \sum_z P\{(X, Z)|Y = 2\} = 1$. We also have:

$$E[X|Y = 2] = 1 \cdot \frac{1}{5} + 2 \cdot (0 + \frac{4}{5}) = \frac{9}{5}$$

Finally, working as above, we get

$$E[X|Y = 2, Z = 1] = 1.$$

3. **(Ross 3.8)**
(a)

$$E[X] = E[X|\text{first roll is } 6]\frac{1}{6} + E[X|\text{first roll is not } 6]\frac{5}{6} = \frac{1}{6} + (1+E[X])\frac{5}{6}$$

implying that $E[X] = 6$ (which is also what you would get by noticing that X follows a *geometric* distribution with success probability $p = 1/6$).

(b)

$$E[X|Y = 1] = 1 + E[X] = 7.$$

(c)

$$\begin{aligned} E[X|Y = 5] &= E[X|Y = 5, X \leq 4]P\{X \leq 4|Y = 5\} + \\ &\quad E[X|Y = 5, X \geq 6]P\{X \geq 6|Y = 5\} \end{aligned} \tag{1}$$

But

$$P\{X \geq 6|Y = 5\} = \frac{P\{X \geq 6, Y = 5\}}{P\{Y = 5\}} =$$

$$\frac{P\{\text{The first four outcomes are different from 5 and 6, and the fifth is equal to 6}\}}{P\{\text{The first four outcomes are different from 5, and the fifth is equal to 5}\}} =$$

$$\frac{(4/6)^4(1/6)}{(5/6)^4 1/6} = (4/5)^4 = 0.4096$$

Also,

$$E[X|Y = 5, X \geq 6] = 5 + E[X] = 5 + 6 = 11$$

On the other hand,

$$E[X|Y = 5, X \leq 4]P\{X \leq 4|Y = 5\} = P\{X \leq 4|Y = 5\}\sum_{i=1}^{4} i \cdot P\{X = i|Y = 5, X \leq 4\} =$$

$$P\{X \leq 4|Y = 5\}\sum_{i=1}^{4} i \cdot \frac{P\{X = i, X \leq 4|Y = 5\}}{P\{X \leq 4|Y = 5\}} = \sum_{i=1}^{4} i \cdot P\{X = i, X \leq 4|Y = 5\} =$$

$$1\left[\frac{1}{5}\right] + 2\left[\frac{4}{5}\right]\left[\frac{1}{5}\right] + 3\left[\frac{4}{5}\right]^2\left[\frac{1}{5}\right] + 4\left[\frac{4}{5}\right]^3\left[\frac{1}{5}\right] = 1.3136$$

Plugging the above values in Eq. (1), we get

$$E[X|Y = 5] = 1.3136 + 11 \cdot 0.4096 \approx 5.8192$$

4. **(Ross 3.9)**

$$E[X|Y=y] = \sum_x xP\{X=x|Y=y\} = \sum_x xP\{X=x\} = E[X].$$

The second equation above holds from the independence of X and Y.

5. **(Ross 3.11)**

$$E[X|Y=y] = C\int_{-y}^{y} x(y^2-x^2)dx = 0.$$

where $C = e^{-y}/8$.

6. **(Ross 3.17)**
Let $K = 1/P\{X=i\}$. Then, by recognizing that $f_{Y|X}(y|i)dy = P\{y \le Y \le y+dy|X=i\}$ and $f_Y(y)dy = P\{y \le Y \le y+dy\}$, and applying the basic definition of conditional probability, we get that

$$f_{Y|X}(y|i) = KP\{X=i|Y=y\}f_Y(y) = K_1 e^{-y}y^i e^{-\alpha y}y^{s-1} = K_1 e^{-(1+\alpha)y}y^{s+i-1}$$

where $K_1 = K\cdot C/i!$ does not depend on y. But as the preceding is the density function of a gamma random variable with parameters $(s+i, 1+\alpha)$ the result follows.

7. **(Ross 3.24)**
In all parts, let X denote the random variable whose expectation is desired, and start by conditioning on the result of the first flip. Also, h stands for heads and t for tails.
(a)

$$E[X] = E[X|h]p + E[X|t](1-p) = (1+\frac{1}{1-p})p + (1+\frac{1}{p})(1-p) = 1+\frac{p}{1-p}+\frac{1-p}{p}$$

(b)

$$E[X] = (1+E[\text{number of heads before first tail}])p + 1(1-p) = 1+p(\frac{1}{1-p}-1) = 1-p+\frac{p}{1-p}$$

(c)
Interchanging p and $1-p$ in (b) gives:

$$E[X] = 1-(1-p)+\frac{1-p}{p}$$

(d)

$$E[X] = (1+\text{answer from (a)})p + (1+\frac{2}{p})(1-p) = (2+\frac{p}{1-p}+\frac{1-p}{p})p + (1+\frac{2}{p})(1-p)$$

8. **(Ross 3.26)**
Let N_A and N_B denote the number of games needed given that you started with A and given that you started with B. Conditioning on the outcome of the first games gives

$$E[N_A] = E[N_A|\omega]p_A + E[N_A|l](1 - p_A)$$

Conditioning on the outcome of the next game gives

$$E[N_A|\omega] = E[N_A|\omega\omega]p_B + E[N_A|\omega l](1-p_B) = 2p_B + (2+E[N_A])(1-p_B) = 2+(1-p_B)E[N_A]$$

As, $E[N_A|l] = 1 + E[N_B]$, we obtain that

$$E[N_A] = (2+(1-p_B)E[N_A])p_A + (1+E[N_B])(1-p_A) = 1+p_A+p_A(1-p_B)E[N_A]+(1-p_A)E[N_B]$$

By symmetry, we have

$$E[N_B] = 1 + p_B + p_B(1 - p_A)E[N_B] + (1 - p_B)E[N_A]$$

Subtracting gives

$$E[N_A]-E[N_B] = p_A - p_B + (p_A - 1)(1-p_B)E[N_A] + (1-p_B)(1-p_A)E[N_B]$$

or equivalently

$$[1 + (1 - p_A)(1 - p_B)](E[N_A] - E[N_B]) = p_A - p_B$$

Since $1 + (1 - p_A)(1 - p_B) > 0$, $p_B > p_A$ implies that $E[N_A] < E[N_B]$, i.e., playing A first is better.

9. **(Ross 3.30)**

$$E[N] = \Sigma_{j=1}^{m} E[N|X_0 = j]p(j) = \Sigma_{j=1}^{m} \frac{1}{p(j)} p(j) = m$$

10. **(Ross 3.31)**
Let L_i denote the length of run i. Conditioning on X, the initial value gives

$$E[L_1] = E[L_1|X = 1]p + E[L_1|X = 0](1-p) = \frac{1}{1-p}p + \frac{1}{p}(1-p) = \frac{p}{1-p} + \frac{1-p}{p}$$

and

$$E[L_2] = E[L_2|X = 1]p + E[L_2|X = 0](1-p) = (\frac{1}{p} - 1)p + (\frac{1}{1-p} - 1)(1-p) = 1$$

11. **(Ross 3.32)**
Let T be the number of trials needed for both at least n successes and m failures. Conditioning on N, the number of successes in the first $n+m$ trials, we obtain that

$$E[T] = \Sigma_{i=0}^{n+m} E[T|N=i] \begin{pmatrix} n+m \\ i \end{pmatrix} p^i (1-p)^{n+m-i}$$

But for $i \leq n$

$$E[T|N=i] = n+m+\frac{n-i}{p}$$

since it will take, on average, $\frac{n-i}{p}$ to obtain the remaining $n-i$ successes. Similarly, for $i > n$,

$$E[T|N=i] = n+m+\frac{i-n}{1-p}$$

Plugging the last two expressions in the first one, gives the result for part (a).

For part (b), let S be the number of trials needed for n successes, and let F be the number needed for m failures. Then the random variable considered in this part is expressed as $\min\{S,F\}$, while the random variable T addressed in part (a) is expressed as $\max(S,F)$. It also holds that

$$\min(S,F) + \max(S,F) = S+F$$

Rearranging the terms in the above identity and taking expectations on both sides, yields:

$$E[\min(S,F)] = \frac{n}{p} + \frac{m}{1-p} - E[T]$$

and the result can be obtained by combining the above equation with the result of part (a).

12. **(Ross 3.35)**

$$\begin{aligned} np_1 &= E[X_1] = E[X_1|X_2=0](1-p_2)^n + E[X_1|X_2>0][1-(1-p_2)^n] \\ &= n\frac{p_1}{1-p_2}(1-p_2)^n + E[X_1|X_2>0][1-(1-p_2)^n] \end{aligned}$$

yielding the result

$$E[X_1|X_2>0] = \frac{np_1(1-(1-p_2)^{n-1})}{1-(1-p_2)^n}$$

13. **(Ross 3.38)**
Let X be the number of successes in the n trials. Now, given that $U = u$, X is binomial with parameters (n, u). As a result,

$$E[X|U] = nU$$

$$E[X^2|U] = n^2U^2 + nU(1-U) = nU + (n^2 - n)U^2$$

Hence

$$E[X] = nE[U] = n/2$$

and

$$E[X^2] = E[nU+(n^2-n)U^2] = nE[U]+(n^2-n)E[U^2] = n/2+(n^2-n)[(1/2)^2+1/12] = n/6+n^2/3$$

Hence,

$$Var(X) = E[X^2] - (E[X])^2 = n/6 + n^2/12$$

14. **(Ross 3.39)**
Let N denote the number of cycles, and let X be the position of card 1.
(a)

$$m_n = \frac{1}{n}\sum_{i=1}^{n} E[N|X=i] = \frac{1}{n}\sum_{i=1}^{n}(1+m_{n-i}) = 1 + \frac{1}{n}\sum_{j=1}^{n-1} m_j$$

where in the last expression we have recognized that $m_0 = 0$.
(b)

$$m_1 = 1$$

$$m_2 = 1 + 1/2 = 3/2$$

$$m_3 = 1 + \frac{1}{3}(1 + 3/2) = 1 + 1/2 + 1/3 = 11/6$$

$$m_4 = 1 + \frac{1}{4}(1 + 3/2 + 11/6) = 25/12$$

(c) Looking at the above expressions for m_1, m_2 and m_3, we are prone to conjecture

$$m_n = 1 + 1/2 + 1/3 + \cdots + 1/n$$

This conjecture is also supported by the derived value for m_4, since

$$1 + 1/2 + 1/3 + 1/4 = 25/12$$

(d)
Obviously, our conjecture holds for $n = 1$. Let us assume that it holds for

all integers up to $n-1$. Then, using the recursion derived in part (a) and the induction hypothesis, we get that

$$m_n = 1+\frac{1}{n}\sum_{j=1}^{n-1}(1+\cdots+1/j) = 1+\frac{1}{n}[n-1+(n-2)/2+(n-3)/3+\cdots+1/(n-1)]$$

But in the above expression, each of the terms $(n-i)/i$, for $i = 1, \ldots, n-1$, can be rewritten as $\frac{n}{i} - 1$, and therefore, we get:

$$m_n = 1 + \frac{1}{n}[n + \frac{n}{2} + \cdots + \frac{n}{n-1} - (n-1)] = 1 + 1/2 + \cdots + 1/n$$

(e)

$$N = \sum_{i=1}^{n} X_i$$

(f)

$$m_n = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P\{\text{i is last of 1,...,i}\} = \sum_{i=1}^{n} 1/i$$

(g) Yes, knowing for instance that $i+1$ is the last of all the cards $1, \ldots, i+1$ to be seen, tells us nothing about whether i is the last of $1, \ldots, i$.

(h)

$$Var(N) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n}(1/i)(1-1/i)$$

where we have used the result that the variance of an indicator variable I_E for some event E is equal to $P(E)(1-P(E))$ (this can be easily shown by noticing that: $Var(I_E) = E[I_E^2] - (E[I_E])^2 = P(E) - (P(E))^2$).

Remark: A simpler way to derive the results for parts (b)–(d) is by using the following, alternative recursion for part (a):

Instead of conditioning on the position of card 1, we condition on the position of card n. Then, letting N denote the number of cycles and X' denote the position of card n, we get

$$m_n = E[N|X' \neq n]P\{X' \neq n\}+E[N|X' = n]P\{X' = n\} = m_{n-1}\frac{n-1}{n}+(1+m_{n-1})\frac{1}{n} = m_{n-1}+\frac{1}{n}$$

Then all the above results follow very naturally when recognizing that $m_0 = 0$.

15. **(Ross 3.42)**
Let X be the number of people who arrive before you. Because you are equally likely to be the first, or second, or third, ..., or eleventh arrival

$$P\{X = i\} = \frac{1}{11},\ i = 0, \ldots 10$$

Therefore,

$$E[X] = \frac{1}{11}\sum_{i=1}^{10} i = \frac{1}{11}\frac{10 \cdot 11}{2} = 5$$

and

$$E[X^2] = \frac{1}{11}\sum_{i=1}^{10} i^2 = \frac{1}{11}\frac{10(11)(21)}{6} = \frac{5 \cdot 21}{3}$$

giving that

$$Var(X) = \frac{5 \cdot 21}{3} - 5^2 = 5(\frac{21}{3} - 5) = 10$$

Another way to derive the above results is as follows:

Let I_i be the indicator variable corresponding to the event that the i-th of the remaining 10 guests arrived before you. Then, the number of guests arriving before you can be expressed as

$$X = \sum_{i=1}^{10} I_i$$

and therefore,

$$E[X] = \sum_{i=1}^{10} E[I_i] = \sum_{i=1}^{10} P\{i\text{-th customer arrived before you}\} = 10\frac{1}{2} = 5$$

Similarly,

$$Var[X] = \sum_{i=1}^{10} Var[I_i] + \sum_{i=1}^{10}\sum_{j \neq i} Cov[I_i, I_j]$$

But from the above discussion on Problem 3.39(h), we have that:

$$Var[I_i] = E[I_i](1 - E[I_i]) = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$$

Also, for $i \neq j$, we have that

$$Cov[I_i, I_j] = E[I_iI_j] - E[I_i]E[I_j] = P\{I_i = 1, I_j = 1\} - (1/2)^2$$

To compute $P\{I_i = 1, I_j = 1\}$, let $X_i,\ X_j$ and X_0 respectively denote the arrival times for guests i, j and yourself. Then,

$$P\{I_i = 1, I_j = 1\} = P\{X_i < X_0 \wedge X_j < X_0\} = \int_0^1 P\{X_i < y\}P\{X_j < y\}f_{X_0}(y)dy = \int_0^1 y^2 dy = 1/3$$

where in the previous calculations we have used the independence of the arrival times and their uniform distribution over the interval $(0, 1)$. The last result implies that

$$Cov[I_i, I_j] = 1/3 - 1/4 = 1/12$$

and plugging the values derived above in the expression for Var[X], we get:

$$Var[X] = 10 \cdot \frac{1}{4} + 10 \cdot 9 \cdot \frac{1}{12} = 10$$

16. **(Ross 3.92)**
Let N denote the number of coins encountered by Josh on his way to work, and denote by X_i the value of the i-th coin, as perceived by Josh (i.e., a penny has *zero* value). Then, the total value collected by Josh on his way to work is expressed as

$$V = \sum_{i=1}^{N} X_i$$

which is a *compound Poisson* r.v. Hence,

$$E[V] = E[N] \cdot E[X_1] = 6\frac{0 + 5 + 10 + 25}{4} = 60$$

and

$$Var[V] = E[N] \cdot E[X_1^2] = 6\frac{25 + 100 + 625}{4} = 1125$$

For part (c), observe that a value $V = 25$ can be obtained by encountering an arbitrary number of pennies *plus:* (i) a quarter; (ii) one nickel and two dimes; (iii) three nickels and one dime; (iv) five nickels. Then, the probability of the considered event can be obtained by characterizing and summing the probabilities of the four events enumerated above. Letting N_1, N_5, N_{10} and N_{25} respectively denote the number of pennies, nickels, dimes and quarters encountered by Josh, we can write the probability of the first of the above events as

$$\sum_{n=1}^{\infty} P\{N_1 = n-1, N_5 = 0, N_{10} = 0, N_{25} = 1 | N = n\} P\{N = n\} = \sum_{n=1}^{\infty} n(\frac{1}{4})^n \frac{e^{-6}6^n}{n!}$$

Similarly,

$$\sum_{n=3}^{\infty} P\{N_1 = n-3, N_5 = 1, N_{10} = 2, N_{25} = 0 | N = n\} P\{N = n\} = \sum_{n=3}^{\infty} \frac{n(n-1)(n-2)}{2}(\frac{1}{4})^n \frac{e^{-6}6^n}{n!}$$

$$\sum_{n=4}^{\infty} P\{N_1 = n-4, N_5 = 3, N_{10} = 1, N_{25} = 0 | N = n\} P\{N = n\} =$$

$$\sum_{n=4}^{\infty} \frac{n(n-1)(n-2)(n-3)}{3!}(\frac{1}{4})^n \frac{e^{-6}6^n}{n!}$$

and

$$\sum_{n=5}^{\infty} P\{N_1 = n-5, N_5 = 5, N_{10} = 0, N_{25} = 0 | N = n\} P\{N = n\} =$$

$$\sum_{n=5}^{\infty} \frac{n(n-1)(n-2)(n-3)(n-4)}{5!} (\frac{1}{4})^n \frac{e^{-6}6^n}{n!} = \sum_{n=5}^{\infty} \left(\begin{array}{c} \text{n} \\ 5 \end{array} \right) (\frac{1}{4})^n \frac{e^{-6}6^n}{n!}$$

Finally, we demonstrate how to calculate the above infinite sums by working with the first of them. The other three can be priced in the same manner. We have:

$$\sum_{n=1}^{\infty} n(\frac{1}{4})^n \frac{e^{-6}6^n}{n!} = \frac{6e^{-9/2}}{4} \sum_{n=1}^{\infty} \frac{e^{-6/4}(6/4)^{n-1}}{(n-1)!} = \frac{6e^{-9/2}}{4} \sum_{n=0}^{\infty} \frac{e^{-6/4}(6/4)^n}{n!} = \frac{6e^{-9/2}}{4}$$

since the last sum is the sum of the pmf of a Poisson random variable with rate $\lambda_1 = 6/4$.

17. **(Ross 3.28)**

Let Y_i be the indicator variable indicating that the i-th selection is red. Then

$$E[X_k] = \sum_{i=1}^{k} E[Y_i]$$

and

$$E[Y_1] = E[X_1] = \frac{r}{r+b}$$

Also,

$$E[Y_2] = E[E[Y_2|X_1]] = E[\frac{r+mX_1}{r+b+m}] = \frac{r}{r+b+m} + \frac{m}{r+b+m} E[X_1] =$$
$$\frac{r}{r+b+m} + + \frac{m}{r+b+m} \frac{r}{r+b} = \frac{r}{r+b}$$

and therefore,

$$E[X_2] = 2\frac{r}{r+b}$$

To prove by induction that $E[Y_k] = \frac{r}{r+b}$, assume that it holds for all $i < k$, and then, we have that:

$$E[Y_k] = E[Y_k|X_{k-1}]] = E[\frac{r+mX_{k-1}}{r+b+(k-1)m}] = \frac{r}{r+b+(k-1)m} + \frac{m}{r+b+(k-1)m} E[\sum_{i=1}^{k-1} Y_i] =$$
$$= \frac{r}{r+b+(k-1)m} + \frac{m}{r+b+(k-1)m}(k-1)\frac{r}{r+b} = \frac{r}{r+b}$$

18. **(Ross 3.63)**

(a) Letting S_i denote the event that there is only one type i coupon in our final collection, and following the hint provided in your textbook, we have

$$P\{S_i\} = \sum_{j=0}^{n-1} P\{S_i|T = j\}P\{T = j\} = \frac{1}{n}\sum_{j=0}^{n-1} P\{S_i|T = j\} = \frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{n-j}$$

The final equality above implies that $P\{S_i|T = j\} = 1/(n-j)$, and it can be justified as follows: Given that we have collected j types before collecting type i, after collecting this type there are $n-j-1$ additional types to be collected. Type i will appear only once in our final collection if and only if in the subsequent activity it will (re-)appear only after the missing $n-j-1$ types have been collected. And the probability of this event is $1/(n-j-1)$.

(b) For this part, just notice that if we let I_i be the indicator variable for event S_i, then, the requested expectation is

$$E[\sum_{i=1}^{n} I_i] = n\frac{1}{n}\sum_{j=0}^{n-1}\frac{1}{n-j} = \sum_{k=1}^{n}\frac{1}{k}$$