Homework 2 Solution

1. **(Ross 2.1)**
A characterization of the sample space of this experiment can be naturally obtained as follows: First, we assign some sort of i.d. to each of the 10 balls in the urn, for instance, number them from 1 to 10. Then, it is clear that the possible outcomes of our experiment are defined by the three-element subsets of the set $\{1, 2, \ldots, 10\}$.

   Obviously, the possible values for X are 0, 1, 2.

   Finally, $\Pr\{X = 0\}$ is the combined probability of all the outcomes in the sample space defined above in which the two selected balls are among the seven balls that are not orange. Given that there are $\binom{7}{2}$ such possible outcomes, and each of them occurs with a probability of $1/\binom{10}{2}$, it follows that the sought probability is $\Pr\{X = 0\} = \binom{7}{2}/\binom{10}{2} = \frac{14}{30}$

2. **(Ross 2.6)**
Event E is defined by the outcome (H, H, H, H, H) and $P\{I_E = 1\} = P(E) = p^5$, where p is the possibility of being head.

3. **(Ross 2.12)**
$\binom{5}{4}\left(\frac{1}{3}\right)^4 \frac{2}{3} + \binom{5}{5}\left(\frac{1}{3}\right)^5 = \frac{10+1}{243} = \frac{11}{243}$

4. **(Ross 2.17)**
Since the outcome of each of the n experiments is independent from the outcomes of the other $n-1$ experiments, it follows that the probability that any single sequence of these n experiments will result in x_i outcomes of type i, $i = 1, 2, \ldots, r$, $\sum_{i=1}^{r} x_i = n$, is equal to

$$p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}$$

   But the number of such possible sequences is equal to the number of the ways that we can partition the n positions of the considered sequence of experiments into r subsets, with subset i consisting of x_i positions, $i = 1, \ldots, r$; hence, this number is equal to

$$\frac{n!}{x_1! \cdots x_r!}$$

   Combining the two expressions above, provides the requested expression.

5. **(Ross 2.18)**
Just substitute $r = 2$ in the result of Problem 2.17.

6. **(Ross 2.19)**
$\Pr\{X_1 + \cdots + X_k = m\} =$
$\Pr\{m$ out of the n experiments result in the first k possible outcomes$\} =$

$$\left[\begin{array}{c} n \\ m \end{array}\right] (p_1 + \cdots + p_k)^m (p_{k+1} + \cdots + p_r)^{n-m}$$

7. **(Ross 2.20)**
This is essentially an application of the result of Problem 2.17 above. $\frac{5!}{2!1!2!}(\frac{1}{5})^2(\frac{3}{10})^2(\frac{1}{2}) = 0.054$

8. **(Ross 2.21)**
One way to compute this probability is by recognizing that the considered event is complementary to the event that number of people buying a TV set is no more than 2. Hence, we have that
$\Pr\{$store owner sells three or more TV sets$\} = 1 - (\frac{3}{10})^5 - 5(\frac{3}{10})^4(\frac{7}{10})$ - $\left[\begin{array}{c} 5 \\ 2 \end{array}\right] (\frac{3}{10})^3(\frac{7}{10})^2$

9. **(Ross 2.25)**
A total of 7 games will be played only if the first 6 result in 3 wins and 3 losses. Thus, $\Pr\{7\text{games}\} = \left(\begin{array}{c} 6 \\ 3 \end{array}\right) p^3(1-p)^3$
As p(1-p) is non negative, $p^3(1-p)^3$ is maximized when p(1-p) is.
Thus p = $\frac{1}{2}$.

10. **(Ross 2.26)**
(a)
Let X denote the number of games played.
$\Pr\{X = 2\} = p^2 + (1-p)^2$ (i.e., one of the teams wins both of the first two games)
$\Pr\{X = 3\} = 2p(1-p)$ (i.e., each team wins one of the first two games)
$E[X] = 2(p^2 + (1-p)^2) + 3(2p(1-p)) = 2 + 2p(1-p)$.

(b)
$\Pr\{X = 3\} = p^3 + (1-p)^3$
$\Pr\{X = 4\} = \Pr\{X = 4,$ team A has 2 wins in first 3 games$\} +$
$\Pr\{X = 4,$ team B has 2 wins in first 3 games$\} = 3p^2(1-p)p + 3p(1-p)^2(1-p)$
$\Pr\{X = 5\} = \Pr\{$each team has 2 wins in the first 4 games$\} = 6p^2(1-p)^2$
$E[X] = 3(p^3 + (1-p)^3) + 12(p^2(1-p)p + p(1-p)^2(1-p)) + 30p^2(1-p)^2$.
Differentiating and setting the resulting expression equal to 0 gives p =

0.5 for both (a) and (b).

11. **(Ross 2.34)**
It must be: $c\int_0^2 (4x - 2x^2)dx = 1$
Hence, $c[2x^2 - 2x^3/3]_0^2 = 1 \Longrightarrow 8c/3 = 1 \Longrightarrow c = \frac{3}{8}$
$\Pr\left\{\frac{1}{2} < X < \frac{3}{2}\right\} = \frac{3}{8}\int_{1/2}^{3/2}(4x - 2x^2)dx$
$= \frac{11}{16}$

12. **(Ross 2.37)**
$\Pr\{M \le x\} = \Pr\{max(X_1, \cdots, X_n) \le x\}$
$= \Pr\{X_1 \le x, \cdots, X_n \le x\}$
$= \prod_{i=1}^n \Pr\{X_i \le x\} = x^n.$
$f_M(x) = \frac{d}{dx}\Pr\{M \le x\} = nx^{n-1}.$

13. **(Ross 2.42)**
Let X_i denote the number of additional coupons collected until the collector obtained his $i+1$ type, after he had got his i-th type, $i = 0, 1, \ldots, n-1$. It is easy to see that the X_i are independent *geometric* random variables with respective parameters (i.e., success probabilities) $(n-i)/n, i = 0, 1, ..., n-1$. Therefore,
$E[\sum_{i=0}^{n-1} X_i] = \sum_{i=0}^{n-1} E[X_i] = \sum_{i=0}^{n-1} n/(n-i) = n\sum_{i=1}^n 1/i$

14. **(Ross 2.51)**
$N = \sum_{i=1}^r X_i$ where X_i is the number of flips between the (i-1)th and i-th head. Hence, X_i is geometric with mean 1/p. Thus,
$E[N] = \sum_{i=1}^r E[X_i] = r/p.$

15. **(Ross 2.55)**
$1 = \int_0^a f(x)dx + \int_a^\infty f(x)dx$
$\le \int_0^a cdx + \Pr\{X > a\}$
$\le ac + \Pr\{X > a\}$

16. **(Ross 2.56)**
Let X_i equal 1 if there is a type i coupon in the collection, and let it be 0 otherwise. The number of distinct types is X $= \sum_{i=1}^n X_i$.
$E[X] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n \Pr\{X_i = 1\} = \sum_{i=1}^n (1 - (1-p_i)^k)$ To compute $Cov(X_i, X_j)$ when i $\neq$ j, note that X_iX_j can take only the values of 0 and 1, and it will be equal 0 if there is no type i or type j coupon in the collection. Therefore,
$\Pr\{X_iX_j = 0\} = \Pr\{X_i = 0\} + \Pr\{X_j = 0\} - \Pr\{X_i = X_j = 0\} = (1 - p_i)^k + (1-p_j)^k - (1-p_i-p_j)^k$

Consequently, for $i \neq j$,
$Cov(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = \Pr\{X_i X_j = 1\} - E[X_i]E[X_j]$
$= 1 - [(1-p_i)^k + (1-p_j)^k - (1-p_i-p_j)^k] - (1-(1-p_i)^k)(1-(1-p_j)^k)$
Furthermore, $Var(X_i) = E[X_i^2] - (E[X_i])^2 = (1-p_i)^k[1-(1-p_i)^k]$
Finally, we can get the requested result from the above by noticing that
$Var(X) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} Cov(X_i, X_j)$

17. **(Ross 2.76)**
Since X and Y are independent, we have
$E[XY] = E[X]E[Y] = \mu_x \mu_y$
and
$E[(XY)^2] = E[X^2]E[Y^2] = (\mu_x^2 + \sigma_x^2)(\mu_y^2 + \sigma_y^2)$
The requested result is obtained by noticing that
$Var(XY) = E[(XY)^2] - (E[XY])^2$