Homework 2 Solution

Part A.

Ch 2 – 12

(a)

$$h = 40 \times \frac{0.35}{52} = 0.269$$

b = 65 – 40 = 25

$$G(R^*) = \frac{25}{25+0.269} = 0.9894$$

$Z_{0.9894} \approx 2.3$ → from normal distribution table

$$R^* = 35 + 2.3 \times 10 = 58$$

Thus, order quantity is 58-12=46

(b)

Now b=5

$$G(R^*) = \frac{5}{5+0.269} = 0.9489$$

$Z_{0.9489} \approx 1.63$ → from normal distribution table

$$R^* = 35 + 1.63 \times 10 = 51.3 \approx 51$$

Thus, order quantity is 51-12=39

Ch 2 – 13

(a) *Poisson* Process, since this demand process accumulates the random demand generated by the 15 parallel stations in the assembly stage, and these stations are not synchronized in any way; in fact, each of them has a quite random / highly variable behavior (see also the discussion on the Poisson distribution provided in pgs 98 – 99 in your textbook, and revisit the material from the IE6650 class on the Poisson process and its properties).

(b) Based on the provided information, each station in the assembly stage will produce 3 units per hour, on average, and therefore, it requires 3 chassis units. Since there are 15 such stations, the average hourly demand is 45 chassis units, and therefore, the average demand over the lead time period of 15 minutes is 45 / 4 = 11.25 units.

Since the demand distribution is Poisson, the above result also translates to a variance of 11.25 units and a standard deviation of 3.3541.

(c) When viewed from the standpoint of the assembly stage, the considered system is essentially a basestock inventory model with its lead time demand distribution characterized in parts (a) and (b) above, and its basestock level R determined by the number of paper cards (essentially KANBANS) that control the material flow from the chassis stage to the assembly stage. Hence, our main problem here is the determination of the minimal basestock level that will guarantee the required service level (fill rate). We know that in the case of discrete distributions G(), the fill rate resulting from any given basestock level R is equal to G(R-1) = G(r), where r is the implied reorder point. Letting r = m, we are looking for the minimal m such that

$$G(m) = \text{Poisson}\ (m; 11.25) \geq 0.99$$

➔ m=20 and the KANBAN level is R=m+1 = 21.

Ch 2 – 17

17. Formulae for some of the quantities:

$\theta = Dl$ $\qquad\qquad$ $\sigma = \sqrt{\theta}$ (because demand is POISSON)

$$F = \frac{D}{Q}$$

$$I(Q,r)*c = [(Q+1)/2 + r - \theta + B(Q,r)]*c$$

Holding cost per year = 12*hI   Order cost per year = 12*FA

The fill rates table is at the end of this problem's solution.

(a,b)

i	c ($/unit)	D_i (units/mo)	l_i (mos)	θ_i (units)	σ_i (units)	Q_i (units)	r_i (units)	F_i (order freq)	Type 1 S	S_i (fill rate)	B_i (backorder	I_i (inventory i
approx	12	15	0.5	7.5	2.7	25	11	0.6	0.92	0.994	0.006	198.07
exact	12	15	0.5	7.5	2.7	25	6	0.6	0.378	0.922	0.133	139.6

As we observed in class, Type 1 service specifications are most stringent than the corresponding Type 2 ones. Therefore, when such a specification is used as an approximation for fill rate, which is another term for the Type 2 service level, it will underestimate its true value, leading to a much larger r and higher inventory.

(c)

i	c ($/unit)	D_i (units/mo)	l_i (mos)	θ_i (units)	σ_i (units)	Q_i (units)	r_i (units)	F_i (order freq)	Type 2 S	S_i (fill rate)	B_i (backorder	I_i (inventory i
approx	12	15	0.5	7.5	2.7	25	6	0.6	0.922	0.922	0.133	139.6
exact	12	15	0.5	7.5	2.7	25	6	0.6	0.922	0.922	0.133	139.6

This approximation is very accurate because it is based on the actual formula that characterizes fill rate, and when Q is this large, the dropped term B(r+Q) is negligible.

(d) i	c ($/unit)	D_i (units/mo)	l_i (mos)	θ_i (units)	σ_i (units)	Q_i (units)	r_i (units)	F_i (order freq)	S_i (fill rate)	B_i (backord	I_i (inventory i
exact	12	15	0.5	7.5	2.7	13	8	1.2	0.934	0.087	91.05

Note that when Q is reduced, we get slightly higher service at a much smaller inventory investment. But of course, we order twice as often. If we neglect the cost or capacity considerations of placing orders, we can always minimize inventory costs y choosing Q=1. But if we consider either order frequency (capacity) or fixed order cost, then EOQ may give a perfectly reasonable Q.

Formulae used in the fill rates table:

$p(r) = \theta^r e^{-\theta}/r!$ (cdf of Poisson random variable)

$G(r) = \sum_{k=0}^{r} p(k)$ (by def on pg. 69 of the textbook)

$B(r) = \theta p(r) + \{[\theta - r]\,[1-G(r)]\}$ (eqn 2.63 on pg. 100. This is the backorder level formula for the base stock model. The values of B(r) are computed because they are used in the following B(Q,r) formula, which is a (Q,r) model formula.)

$B(Q, r) = \frac{1}{Q} \sum_{x=r+1}^{r+Q} B(x)$ (eqn 2.38 on pg. 78)

Type 1 service = G (r)   (eqn 2.36 on pg. 78)

Type 2 service = $1 - \frac{B(r)}{Q}$ (eqn 2.37 on pg. 79)

Exact S(Q,r) = $1 - \frac{1}{Q}[B(r) - B(r+Q)]$ (eqn 2.35 on pg. 78)

Fill rates table for Problem 2.17:

θ = 7.5

r	p(r)	G(r)	B(r)	Q= 25			Q= 13	
		Type 1 S		B(Q,r)	Type 2 S	Exact S	B(Q,r)	Exact S
0	0.001	0.001	7.500	1.125	0.700	0.700	2.161	0.426
1	0.004	0.005	6.501	0.865	0.740	0.740	1.662	0.501
2	0.016	0.020	5.505	0.645	0.780	0.780	1.240	0.577
3	0.039	0.059	4.526	0.464	0.819	0.819	0.892	0.652
4	0.073	0.132	3.585	0.320	0.857	0.857	0.616	0.724
5	0.109	0.241	2.717	0.212	0.891	0.891	0.407	0.791
6	0.137	**0.378**	1.958	**0.133**	**0.922**	**0.922**	0.256	0.849
7	0.146	0.525	1.336	0.080	0.947	0.947	0.154	0.897
8	0.137	0.662	0.861	0.045	0.966	0.966	**0.087**	**0.934**
9	0.114	0.776	0.523	0.025	0.979	0.979	0.047	0.960
10	0.086	0.862	0.299	0.013	0.988	0.988	0.024	0.977
11	0.059	**0.921**	0.162	**0.006**	0.994	**0.994**	0.012	0.988
12	0.037	0.957	0.082	0.003	0.997	0.997	0.005	0.994
13	0.021	0.978	0.040	0.001	0.998	0.998	0.002	0.997
14	0.011	0.990	0.018	0.001	0.999	0.999	0.001	0.999
15	0.006	0.995	0.008	0.000	1.000	1.000	0.000	0.999
16	0.003	0.998	0.003	0.000	1.000	1.000	0.000	1.000
17	0.001	0.999	0.001	0.000	1.000	1.000	0.000	1.000
18	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
19	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
20	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
21	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
22	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
23	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
24	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
25	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
26	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
27	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
28	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
29	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
30	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
31	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
32	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
33	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
34	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000
35	0.000	1.000	0.000	0.000	1.000	1.000	0.000	1.000

Part B.

Problem 1

i. <u>Solution Methodology</u>

In this multi-product newsboy problem, the objective is to find order quantities Q_i for newspaper i, i=1,…,N, such that the daily profit is maximized while the total weight of newspapers $\sum_{i=1}^{N} Q_i w_i$ is less than or equal to W.

Maximizing the profit is equivalent to minimizing the total cost, which consists of the underage and overage costs. Let $u_i = p_i - c_i$ be the unit underage cost and $o_i = c_i - s_i$ be the unit overage cost for newspaper i. Then the cost contributed by product i is

$$C_i (Q_i, X_i) = o_i \max\{0, Q_i - X_i\} + u_i \max\{0, X_i - Q_i\}$$

Taking the expected value of the cost with respect to the demand X_i and summing up for the N products, the optimization problem (P) can be formulated as

Minimize

$$\sum_{i=1}^{N} \left[o_i \int_0^{Q_i} (Q_i - x) g_i(x) dx + u_i \int_{Q_i}^{\infty} (x - Q_i) g_i(x) dx \right]$$

subject to

$$\sum_{i=1}^{N} Q_i w_i \le W$$

where $g_i(x)$ is the probability density function of the demand X_i.

Note that in the basic newsboy model, where there is no weight constraint, $Q_i = G_i^{-1}\left(\frac{u_i}{o_i + u_i}\right)$ gives separate optimal ordering quantity for newspaper i. So if the combination of these quantities does not violate the weight constraint, i.e. $\sum_{i=1}^{N} G_i^{-1}\left(\frac{u_i}{o_i + u_i}\right) w_i \le W$, then it is a feasible and optimal solution to (P). If $\sum_{i=1}^{N} G_i^{-1}\left(\frac{u_i}{o_i + u_i}\right) w_i > W$, then the optimal solution will satisfy the weight constraint as equality. The reason is that the expected cost of each newspaper, $E[C_i (Q_i, X_i)] = o_i \int_0^{Q_i} (Q_i - x) g_i(x) dx + u_i \int_{Q_i}^{\infty} (x - Q_i) g_i(x) dx$, is a convex function in Q_i. For

$0 \le Q_i < G_i^{-1}\left(\frac{u_i}{o_i + u_i}\right)$, E[$C_i$ (Q_i, X_i)] is decreasing in Q_i. For any solution that gives a total weight strictly less than W, the objective function can be improved by increasing some of the Q_i's until the weight constraint is satisfied at equality.

Assuming $\sum_{i=1}^{N} G_i^{-1}\left(\frac{u_i}{o_i + u_i}\right) w_i > W$, we can replace the inequality sign with equality in the constraint in (P) and obtain the same optimal solutions. In this case we may introduce a Lagrange multiplier θ and find the optimal solution to (P) by solving the unconstrained problem:

Minimize

$$C(Q_1,...,Q_N,\theta) = \sum_{i=1}^{N}\left[o_i \int_0^{Q_i}(Q_i - x)g_i(x)dx + u_i \int_{Q_i}^{\infty}(x - Q_i)g_i(x)dx\right] + \theta\left(\sum_{i=1}^{N} w_i Q_i - W\right)$$

The optimality conditions are:

$$\frac{\partial C}{\partial Q_i} = o_i G_i(Q_i) - u_i + u_i G_i(Q_i) + \theta w_i = 0, \forall i = 1,...,N \qquad \text{...... (1)}$$

and

$$\frac{\partial C}{\partial \theta} = \sum_{i=1}^{N} w_i Q_i - W = 0 \qquad \text{...... (2)}$$

From (1), we have $G_i(Q_i) = \frac{u_i - \theta w_i}{o_i + u_i}$, so we can write Q_i in terms of θ as

$$Q_i^*(\theta) = G_i^{-1}\left(\frac{u_i - \theta w_i}{o_i + u_i}\right).$$

Then the problem becomes finding a value of θ such that $Q_i^*(\theta)$ satisfies (2). θ can be solved using bisection search over the interval between a lower bound and upper bound for θ. Note that θ can be interpreted as the penalty cost of violating the weight constraint by one unit, so a lower bound for θ is 0. Also, since G_i is a cumulative distribution function, θ has to satisfy $0 \le \frac{u_i - \theta w_i}{o_i + u_i} \le 1$, i.e. $-\frac{o_i}{w_i} \le \theta \le \frac{u_i}{w_i}$. Therefore, $\min_i \frac{u_i}{w_i}$ can be taken as the initial upper bound for θ in the bisection search.

During bisection search, set θ to be the midpoint between the upper and lower bounds. Stop if $0 \le W - \sum_{i=1}^{N} w_i Q_i(\theta) < \min_i w_i$. $\lfloor Q_i(\theta) \rfloor$ is optimal. If $W - \sum_{i=1}^{N} w_i Q_i \ge \min_i w_i$, replace the upper bound with the current value of θ. If $W - \sum_{i=1}^{N} w_i Q_i < 0$, replace the lower bound with the current value of θ. Repeat until an optimum is found.

ii. Application (**Remark:** Notice that in the following calculations, the notation N(a,b) employed in the problem data, has been interpreted as a normal distribution with mean equal to a and st. deviation equal to b.)

i	u_i	o_i	critical ratio = $u_i/(u_i+o_i)$
1	0.75	0.15	0.83
2	1.00	0.40	0.71
3	1.30	0.60	0.68

Iteration	Lower	Upper	θ	Q1	Q2	Q3	Total Weight
0			0	119.35	80.66	54.80	174.96
1	0	1.3	0.65	98.61	71.58	45.93	148.92
2	0	0.65	0.325	107.86	76.01	50.33	161.26
3	0.325	0.65	0.4875	103.15	73.82	48.18	155.12
4	0.4875	0.65	0.56875	100.87	72.71	47.07	152.04
5	0.56875	0.65	0.609375	99.74	72.15	46.51	150.49
6	0.609375	0.65	0.629688	99.17	71.87	46.22	149.71

At Iteration 6, the difference between the total weight and the allowable weight is 0.29 l b, less than the weight of the lightest paper (paper 1, 0.5 lb), so we stop there. Roundin g down Q_i's to integers, we get Q_1 = 99, Q_2 = 71 and Q_3 = 46. That frees up 1.25 lbs and allows the newsboy to carry an additional copy of Paper 1 and 2 each. The final answer is Q_1 = 100, Q_2 = 72 and Q_3 = 46.

Problem 2

b=100 $/pound.

h=200 $/pound

$G(Q^*)=\frac{100}{100+200} = 0.33$

$Z_{0.33}$ =-0.44 →from normal distribution table

Q*=100,000-0.44*5000=97,800 pounds per month.

Therefore the optimal selection of the plant capacity equals 97,800 pounds per month.

Extra Credit

Problem 1

$$
\begin{aligned}
B(R) \quad &= \sum_{x=R}^{\infty}(x-R)p(x) \\
&= \sum_{x=0}^{\infty}(x-R)p(x) - \sum_{x=0}^{R-1}(x-R)p(x) \\
&= \sum_{x=0}^{\infty} xp(x) - R\sum_{x=0}^{\infty} p(x) - \sum_{x=0}^{R-1}(x-R)p(x) \\
&= \theta - R(1) + \sum_{x=0}^{R-1}(R-x)p(x) \\
(*) \quad &= \theta - R + \sum_{x=0}^{R-1}\sum_{y=0}^{x} p(y) \\
&= \theta - R + \sum_{x=0}^{R-1} G(x) \\
&= \theta - \sum_{x=0}^{R-1}[1-G(x)]
\end{aligned}
$$

(*) follows from the previous line because

$$\begin{aligned}
\sum_{x=0}^{R-1}(R-x)p(x) &= \underbrace{p(0)+p(0)+\ldots+p(0)}_{\text{R times}}+\underbrace{p(1)+\ldots+p(1)}_{\text{R-1 times}}+\ldots+\underbrace{p(R-1)}_{\text{1 time}} \\
&= p(0) \\
&+ p(0)+p(1) \\
&+ \ldots \\
&+ p(0)+p(1)+\ldots+p(R-2) \\
&+ p(0)+p(1)+\ldots+p(R-2)+p(R-1) \\
&= \sum_{x=0}^{R-1}\sum_{y=0}^{x} p(y)
\end{aligned}$$

Problem 2

2. Show that $Var[X]=E[L]\cdot Var[D]+E[D]^2\cdot Var[L]$

Proof:

$$\begin{aligned}
&Var(X) \\
&= E(X^2)-[E(X)]^2 \\
&= E[E(X^2\mid L)]-[E(E[X\mid L])]^2 \\
&= E[E(X^2\mid L)]-E[(E[X\mid L])^2]+Var(E[X\mid L]) \\
&= E[E(X^2\mid L)-(E[X\mid L])^2]+Var(E[X\mid L]) \\
&= E[Var(X\mid L)]+Var(E[X\mid L])
\end{aligned}$$

$$\begin{aligned}
&Var(X\mid L=n) \\
&= Var\left(\sum_{t=1}^{L} D_t \middle| L=n\right) \\
&= Var\left(\sum_{t=1}^{n} D_t \middle| L=n\right) \\
&= Var\left(\sum_{t=1}^{n} D_t\right) && \text{(L and } D_t \text{ are independent)} \\
&= \sum_{t=1}^{n} Var[D_t] && (D_t \text{ are independent)} \\
&= nVar[D] \\
&\Rightarrow E[Var(X\mid L)] = E[L\cdot Var(D)] = E[L]\cdot Var(D)
\end{aligned}$$

Also,

$$Var(E[X \mid L])$$
$$= Var(L \cdot E[D])$$
$$= E[D]^2 \cdot Var(L)$$

Therefore, we get $Var[X] = E[L] \cdot Var[D] + E[D]^2 \cdot Var[L]$.

Problem 3

Let g(x) be the distribution function of the random demand X in the newvendor problem. Then

$$
\begin{aligned}
&(c-s)E[\max\{Q-X,0\}]+(p-c)E[\max\{X-Q,0\}]\\
&=c\int_0^Q (Q-x)g(x)dx - sE[\max\{Q-X,0\}]+(p-c)\int_Q^\infty (x-Q)g(x)dx\\
&=c\int_0^\infty (Q-x)g(x)dx - sE[\max\{Q-X,0\}]+p\int_Q^\infty (x-Q)g(x)dx\\
&=cQ\int_0^\infty g(x)dx - c\int_0^\infty xg(x)dx - sE[\max\{Q-X,0\}]+p\left[\int_Q^\infty xg(x)dx - \int_Q^\infty Qg(x)dx\right]\\
&=cQ-cE[X]-sE[\max\{Q-X,0\}]+p\left[\int_Q^\infty xg(x)dx + \int_0^Q xg(x)dx - \int_0^Q xg(x)dx - \int_Q^\infty Qg(x)dx\right]\\
&=cQ-cE[X]-sE[\max\{Q-X,0\}]+pE[X]-pE[\min\{Q,X\}]\\
&=-\{pE[\min\{Q,X\}]+sE[\max\{Q-X,0\}]-cQ\}+(p-c)E[X] \qquad \text{(Equation 1)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\min\ (c-s)E[\max\{Q-X,0\}]+(p-c)E[\max\{X-Q,0\}]\\
\Leftrightarrow\ &\min\ -\{pE[\min\{Q,X\}]+sE[\max\{Q-X,0\}]-cQ\}+(p-c)E[X]\\
\Leftrightarrow\ &\max\ pE[\min\{Q,X\}]+sE[\max\{Q-X,0\}]-cQ-(p-c)E[X]
\end{aligned}
$$

Note that the last term $(p-c)E[X]$ can be dropped from the objective function because it is independent of Q. Therefore, the two objectives are equivalent.

Remark: Maybe a better way to understand the result of Eq. 1, is by re-writing it as:

$$
\begin{aligned}
&pE[\min\{Q,X\}]+sE[\max\{Q-X,0\}]-cQ=\\
&(p-c)E[X]-(p-c)E[\max\{X-Q,0\}]-(c-s)E[\max\{Q-X,0\}]
\end{aligned}
$$