Optimal order sizing for the newsvendor model with discrete demand

Spyros Reveliotis

Consider a newsvendor problem with unit overage cost c_o, unit shortage cost c_s, and the demand taking the discrete values D_i, for $i \in \{1,2,\ldots,n\}$, with corresponding probabilities p_i (the results hold even if $n = \infty$). Also, let $TC(Q)$ denote the expected total cost resulting from some order size Q.

In this document we shall show that an optimal selection for the order size Q is provided by the *smallest* demand level D_i such that $Prob(D \leq D_i) \equiv \sum_{k=1}^{i} p_k$ is greater than or equal to the problem critical ratio $c_s/(c_s+c_o)$.

We shall establish this result in two steps: First we show that there is no advantage in selecting an order size Q that does not coincide with one of the discrete levels of the demand. Once we have narrowed down the choices of Q as stated above, then, we shall prove the main result.

Lemma 1 *Consider an order size Q that belongs in some open interval (D_i, D_{i+1}), i.e., Q is strictly between the demand levels D_i and D_{i+1}. Then,*

$$TC(Q) \geq \min\{TC(D_i), TC(D_{i+1})\}.$$

Proof: Since $Q \in (D_i, D_{i+1})$, we can write $Q = D_i + \alpha$, where $0 < \alpha < D_{i+1} - D_i$. Also, we set $\beta = D_{i+1} - (D_i + \alpha)$. From the above definitions, it is also clear that $\beta > 0$. Furthermore, we can express $TC(Q)$ as follows:

$$\begin{aligned}
TC(Q) &= c_0 \sum_{k=1}^{i} p_k(Q - D_k) + c_s \sum_{k=i+1}^{n} p_k(D_k - Q) \\
&= c_o \sum_{k=1}^{i} p_k(D_i + \alpha - D_k) + c_s \sum_{k=i+1}^{n} p_k(D_k - D_i - \alpha) \\
&= c_o \sum_{k=1}^{i} p_k(D_i - D_k) + c_o \sum_{k=1}^{i} p_k \alpha \\
&\quad + c_s \sum_{k=i+1}^{n} p_k(D_k - D_i) - c_s \sum_{k=i+1}^{n} p_k \alpha \qquad (1)
\end{aligned}$$

We also have:

$$TC(D_i) = c_o \sum_{k=1}^{i} p_k(D_i - D_k) + c_s \sum_{k=i+1}^{n} p_k(D_k - D_i) \qquad (2)$$

From Eqs 1 and 2, we have:

$$TC(Q) - TC(D_i) = \alpha[c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k] \tag{3}$$

So, if $[c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k] \geq 0$, we have $TC(Q) \geq TC(D_i)$, and the lemma holds true.

In the opposite case (i.e., if $c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k < 0$), consider $TC(D_{i+1})$. We have:

$$TC(D_{i+1}) = c_o \sum_{k=1}^{i} p_k(D_{i+1} - D_k) + c_s \sum_{k=i+1}^{n} p_k(D_k - D_{i+1}) \tag{4}$$

Also, from Eqs 1 and 4:

$$\begin{aligned}
TC(Q) - TC(D_{i+1}) &= c_o \sum_{k=1}^{i} p_k(D_i - D_{i+1}) + c_o \sum_{k=1}^{i} p_k \alpha \\
&\quad - c_s \sum_{k=i+1}^{n} p_k \alpha - c_s \sum_{k=i+1}^{n} p_k(D_i - D_{i+1}) \\
&= c_o \sum_{k=1}^{i} p_k(D_i + \alpha - D_{i+1}) \\
&\quad - c_s \sum_{k=i+1}^{n} p_k(D_i + \alpha - D_{i+1}) \\
&= -c_o \sum_{k=1}^{i} p_k \beta + c_s \sum_{k=i+1}^{n} p_k \beta \\
&= -\beta[c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k] > 0 \qquad (5)
\end{aligned}$$

The last inequality in Eq. 5 results from the working hypothesis that $c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k < 0$. Eq. 5 implies that in this second case $TC(Q) > TC(D_{i+1})$, and once again, Lemma 1 is true. Furthermore, since the two considered cases for the sign of the difference $TC(Q) - T(D_i)$ exhaust all the possibilities, Lemma 1 must hold true. $\square$

Next we state and prove the main result of this document.

Theorem 1 *In the considered newsvendor problem, an optimal selection for the order size Q is the* smallest *demand level D_i such that*

$$Prob(D \leq D_i) \ \equiv \ \sum_{k=1}^{i} p_k \ \geq \ \frac{c_s}{c_s + c_o}.$$

Proof: From Lemma 1, we know that we can restrict our search for an optimal selection of Q over the set of the discrete demand levels $D_i,\ i = 1, 2, \ldots, n$. Next, we consider the difference $TC(D_{i+1}) - TC(D_i)$. From Eqs 2 and 4, we have:

$$
\begin{aligned}
TC(D_{i+1}) - TC(D_i) &= c_o \sum_{k=1}^{i} p_k (D_{i+1} - D_k) + c_s \sum_{k=i+1}^{n} p_k (D_k - D_{i+1}) \\
&\quad - c_o \sum_{k=1}^{i} p_k (D_i - D_k) - c_s \sum_{k=i+1}^{n} p_k (D_k - D_i) \\
&= c_o \sum_{k=1}^{i} p_k (D_{i+1} - D_i) - c_s \sum_{k=i+1}^{n} p_k (D_{i+1} - D_i) \\
&= (D_{i+1} - D_i)[c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k] \qquad (6)
\end{aligned}
$$

Eq. 6 further implies that

$$
\begin{aligned}
TC(D_{i+1}) - TC(D_i) < 0 &\iff c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k < 0 \\
&\iff \frac{\sum_{k=1}^{i} p_k}{\sum_{k=i+1}^{n} p_k} < \frac{c_s}{c_o} \\
&\iff \frac{\sum_{k=1}^{i} p_k}{\sum_{k=1}^{i} p_k + \sum_{k=i+1}^{n} p_k} < \frac{c_s}{c_s + c_o} \\
&\iff \sum_{k=1}^{i} p_k < \frac{c_s}{c_s + c_o} \qquad (7)
\end{aligned}
$$

In the last derivation of Eq. 7 we have taken into consideration the fact that $\sum_{k=1}^{i} p_k + \sum_{k=i+1}^{n} p_k = \sum_{k=1}^{n} p_k = 1.0$. In plain terms, Eq. 7 implies that the expected total cost can decrease as we go from some demand level D_i to the next level D_{i+1} if and only if $\sum_{k=1}^{i} p_k < \frac{c_s}{c_s+c_o}$. Hence, we can stop as soon as we reach a demand level $D_{\hat{i}}$ such that $\sum_{k=1}^{\hat{i}} p_k \geq \frac{c_s}{c_s+c_o}$. At that point, we set $Q = D_{\hat{i}}$. $\square$