Chapter 4

Problem 3

$T = S + P + R$
so
$E[T] = E[S] + E[P] + E[R]$
$V[T] = V[S] + V[P] + V[R]$
and $C^2[X] = V[X]/E[X]^2 \quad \Rightarrow V[X] = C^2[X] \times E[X]^2$

(a) let
$E[S] = 10$ min, $\quad C^2[S] = 3,$
$E[P] = 60$ min, $\quad C^2[P] = 1/2,$
$E[R] = 5$ min, $\quad C^2[R] = 1,$
thus
$V[S] = 3(10)^2, \quad V[P] = \frac{1}{2}(60)^2, \quad V[R] = 1(5)^2$
and
$E[T] = 75, \quad V[T] = 2125 \text{ min}^2, \quad C^2[T] = \frac{2125}{75^2} = 0.377\overline{7}$

(b) the parameters are now
$E[S] = 1$ min, $\quad C^2[S] = 1,$
$E[P] = 60$ min, $\quad C^2[P] = 1/2,$
$E[R] = 5$ min, $\quad C^2[R] = 1/3,$
thus
$V[S] = 1(1)^2, \quad V[P] = \frac{1}{2}(60)^2, \quad V[R] = \frac{1}{3}(5)^2$
and
$E[T_{new}] = 66, \quad V[T_{new}] = 1809.33 \text{ min}^2, \quad C^2[T_{new}] = 0.415$

$E[T_{new}] = 12$ % improvement,
$C^2[T_{new}] = -9.95$ % improvement (it went up!)

Problem 4

$a = E[F]/(E[F]+E[R])$, availability fraction for machine

$E[T_e] = E[T]/a$, effective mean processing time

$C^2[T_e] = C^2[T] + \left(1 + C^2[R]\right)a(1-a)E[R]/E[T]$

let

$E[T] = 3, \quad C^2[T] = 2$, normal processing time

$E[F] = 7, \quad C^2[F] = 1$, time between failures (required that $C^2[F] = 1$)

$E[R] = 3, \quad C^2[R] = 2$, repair time.

then

$a = E[F]/(E[F]+E[R]) = 7/(7+1) = 7/8$

$E[T_e] = E[T]/a = 3 \times 8/7 = 3.4286$

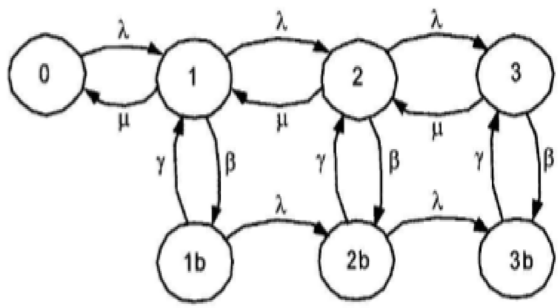
$$C^2[T_e] = C^2[T] + \left(1 + C^2[R]\right)a(1-a)E[R]/E[T]$$

$$= 2 + (1+1)\frac{7}{8} \times \frac{1}{8} \times 1/3 = 2 + 14/192 = 2.0729$$

$$V[T_e] = C^2[T_e] \cdot E^2[T_e] = 2.0729 \times 3.4286^2 =$$

$$= 24.36756 \text{ hrs}^2$$

Problem 7.
This problem can be modeled as an M/M/1/3 with breakdown and repair parameters β and γ and states (0,1,1b,2,2b,3,3b):



$$\begin{aligned}
0: &\quad \mu p_1 - \lambda p_0 = 0 \\
1: &\quad \mu p2 + \lambda p_0 + \gamma p_{1b} - (\lambda + \mu + \beta)p_1 = 0 \\
1b: &\quad \beta p_1 - (\lambda + \gamma)p_{1b} = 0 \\
2: &\quad \mu p3 + \lambda p_1 + \gamma p_{2b} - (\lambda + \mu + \beta)p_2 = 0 \\
2b: &\quad \beta p_2 + \lambda p_{1b} - (\lambda + \gamma)p_{2b} = 0 \\
3: &\quad \lambda p_2 + \gamma p_{3b} - (\mu + \beta)p_3 = 0 \\
&\quad \sum_{i \in \{0,1,1b,2,2b,3,3b\}} p_i = 1
\end{aligned}$$

Solving these equation with $\lambda = 5hr^{-1}, \mu = 4hr^{-1}, \beta = 1hr^{-1}$ and $\gamma = 6hr^{-1}$ we got
$(p_0, p_1, p_{1b}, p_2, p_{2b}, p_3, p_{3b}) : \{0.138, 0.173, 0.16, 0.236, 0.026, 0.33, 0.079\}$

$$\begin{aligned}
\mathsf{WIP}_s &= 1(p_1 + p_{1b}) + 2(p_2 + p_{2b}) + 3(p_3 + p_{3b}) = 1.94 \\
\mathsf{TH}_s &= \lambda(1 - p_3 - p_{3b}) = 2.955 \\
CT_s &= \frac{\mathsf{WIP}_s}{\mathsf{TH}_s} = 0.657 \\
\mathsf{Idle} &= p_0 = 0.138 \\
\mathsf{Down} &= p_{1b} + p_{2b} + p_{3b} = 0.121 \\
\mathsf{Processing} &= p_1 + p_2 + p_3 = 0.739
\end{aligned}$$

Problem B1:

We have:

$r_a = 0.1/\text{min}$

$c_a^2 = 0$

$u = r_a t_p = 0.95$ therefore $t_p = 9.5/\text{min}$

$CT_q = \frac{c_a^2 + c_p^2}{2} \times \frac{u}{1-u} \times t_p = 45$ min therefore $c_p^2 = \frac{45}{\frac{1}{2} \times \frac{0.95}{0.05} \times 9.5} = 0.4986$ and

$c_d^2 = u^2 c_p^2 + (1 - u^2) c_a^2 = 0.95^2 \times 0.4986 = 0.45$

Furthermore, since it is a stable system,

$r_a = r_d$

Finally,

$t_d = \frac{1}{r_a} = 10$ min

$\sigma_d^2 = c_d^2 t_d^2 = 45\ \text{min}^2$

Problem B2:

i. $A = \dfrac{m_f}{m_f + m_r}$, $\quad A_1 = \dfrac{7}{7+1.5} = 0.824$, $\quad A_2 = \dfrac{5}{5+0.5} = 0.909$

$t_e = \dfrac{t_0}{A}$, $\quad t_{e1} = \dfrac{11}{0.824} = 13.36\min = 0.223hr$, $\quad t_{e2} = \dfrac{11}{0.909} = 12.1\min = 0.202hr$

Station 1 is the effective bottleneck of the line.

ii. $u = r_a t_e$, $\quad u_1 = \dfrac{35}{8} \cdot \dfrac{13.36}{60} = 0.974$, $\quad u_2 = \dfrac{35}{8} \cdot \dfrac{12.1}{60} = 0.882$

The utilization is less than 1 at both stations, so the production line can sustain the production rate of 35 parts per 8-hour shift.

iii. For station 1, the SCV of the effective processing time is

$$c_e^2 = c_0^2 + (1 + c_r^2)(1 - A)A\frac{m_r}{t_0} = 0.5^2 + (1 + 0.75^2)(1 - 0.824)(0.824)\frac{1.5}{11/60} = 2.108$$

$$CT = \left(\frac{c_a^2 + c_e^2}{2} \cdot \frac{u}{1-u} + 1\right)t_e = 2 \text{ hr and } 7.815 \text{ min} = 2.13 \text{ hr}$$

$$\Rightarrow \left(\frac{0 + 2.108}{2} \cdot \frac{r_a \cdot 0.223}{1 - r_a \cdot 0.223} + 1\right)0.223 = 2.13$$

$\Rightarrow r_a = 4$ parts per hour

The inter-release interval is 15 minutes.

iv. $CT_q = CT - t_e = 1.908hr$, $\quad WIP_q = TH \cdot CT_q = 4 \times 1.908 = 7.63$

v. Mean = 15 minutes = $\dfrac{1}{4}$ hrs

Variance = variance of inter-departure times

$$= c_{d1}^2 \cdot \left(\frac{1}{4}\right)^2 =$$

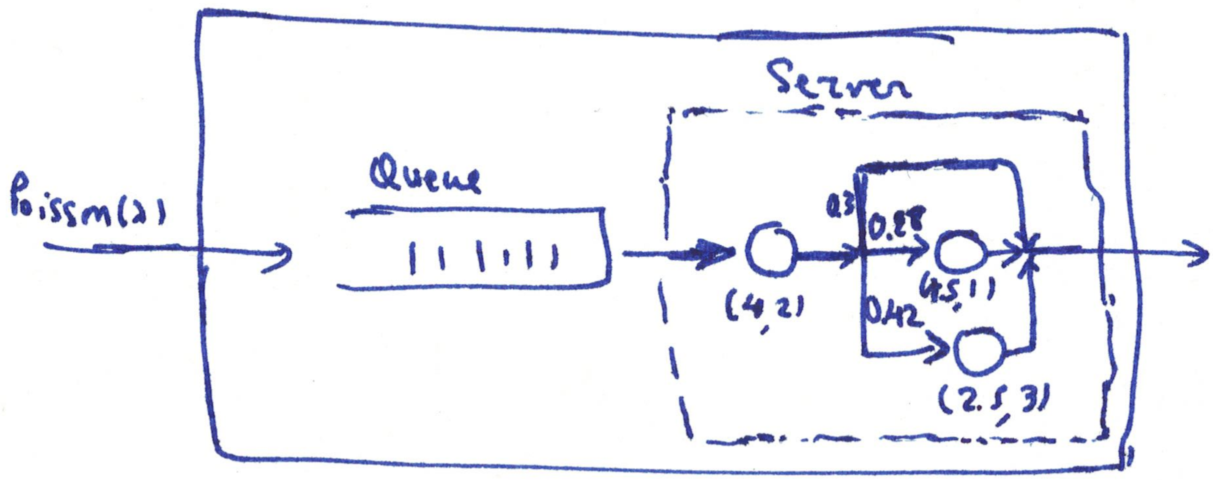
$$\left(u_1^2 c_{e1}^2 + (1 - u_1^2) \cdot c_{a1}^2\right)\left(\frac{1}{4}\right)^2 = \left[(4 \cdot 0.223)^2 \cdot 2.108 + 0\right]\left(\frac{1}{4}\right)^2$$

$= 0.1045$ (hrs)^2 = 376.2 (min)^2

Problem 3:

①

The new situation in the considered workstation can be depicted as follows:



Let r.v. T_e denote the effective proc. time in this new situation. Also let r.v. T_p denote the proc. time for the first proc. stage, and r.v.'s Tr_I and Tr_{II} denote the rework times for type I and type II rework respectively. Then, we have

$$T_e = T_p + X$$

where X is another r.v. that is defined as follows:

$$X \sim \begin{cases} \emptyset & \text{w.p. } 0.3 \\ Tr_I & \text{w.p. } 0.28 \\ Tr_{II} & \text{w.p. } 0.42 \end{cases}$$

Hence,

$$E[X] = 0 \times 0.3 + 1.5 \times 0.28 + 2.5 \times 0.42 = 1.47 \text{ min}$$

Also

$$E[X^2] = 0 \times 0.3 + E[Tr_I^2] \times 0.28 + E[T_{II}^2] \times 0.42 =$$

$$= (Var[Tr_I] + E^2[Tr_I]) \times 0.28 + (Var[Tr_{II}] + E^2[T_{r_I}]) \times 0.42 =$$

$$= (1 + 1.5^2) \times 0.28 + (3^2 + 2.5^2) \times 0.42 = 7.315 \text{ min}^2$$

and $Var[X] = E[x^2] - E^2[x] = 7.315 - 1.47^2 = 5.1541 \text{ min}^2$

Then

$t_e' = E[T_e] = E[T_p] + E[x] = 4 + 1.47 \approx 5.47 \text{ min}$

$Var[T_e] = Var[T_p] + Var[x] = 2^2 + 5.1541 = 9.1541 \text{ min}^2$

$SCV[T_e] = Var[T_e] / E[T_e] = 9.1541 / 5.47^2 \approx 0.3$

We also have $u = \lambda . t_e = \frac{10}{60} \times 5.47 = 0.9117$

and

$CT = \frac{1 + 0.3}{2} \frac{0.9117}{1 - 0.9117} 5.47 + 5.47 = 42.18 \text{ min}$

$CT_q = CT - t_e = 42.18 - 5.47 = 36.71 \text{ min}.$

$WIP = \lambda . CT = \frac{10}{60} \times 42.18 = 7.03$

$WIP_q = \lambda CT_q = WIP - u = 7.03 - 0.9117 = 6.1183$

Extra Credit Problem

To answer the question of this problem, we need to compute the mean effective proc. time that takes into consideration the rework that is performed at this workstation.

We have

$$t_e = E[T_{eff}] = E[T_p + T_{rew}] =$$

$$= E[T_p] + E[T_{rew}] =$$

$$= t + p_1(1-p_2)\frac{1}{r_1} + p_2(1-p_1)\frac{1}{r_2} +$$

$$+ p_1 \cdot p_2 \cdot E\left[\max\left(T_{rew}^{(1)}, T_{rew}^{(2)}\right)\right]$$

In the last expression above:

* t denotes the mean fastening time for two good parts.
* The second term is the contribution to $E[T_{rew}]$ of the case where only the first part needs rework.
* The third term is the contribution to $E[T_{rew}]$ of the case where only the second part needs rework.
* And the last term is the contribution to $E[T_{rew}]$ of the case where both parts need rework.

~~(We must take the max~~

In this case, $T_{rew}^{(i)}$, $i=1,2$, is a r.v. denoting the rework time for part i, and we must take the max of these two times since reworks are performed simultaneously.

To get $E\left[\max\left(T_{rew}^{(1)}, T_{rew}^{(2)}\right)\right]$, we work as follows:

$$E\left[\max\left(T_{rew}^{(1)}, T_{rew}^{(2)}\right)\right] =$$

$$= E\left[\max\left(T_{rew}^{(1)}, T_{rew}^{(2)}\right) \mid T_{rew}^{(1)} < T_{rew}^{(2)}\right] \cdot P\left(T_{rew}^{(1)} < T_{rew}^{(2)}\right) +$$
$$+ E\left[\max\left(T_{rew}^{(1)}, T_{rew}^{(2)}\right) \mid T_{rew}^{(1)} > T_{rew}^{(2)}\right] \cdot P\left[T_{rew}^{(1)} > T_{rew}^{(2)}\right)$$

$$= E\left[T_{rew}^{(1)} + \left(T_{rew}^{(2)} - T_{rew}^{(1)}\right) \mid T_{rew}^{(1)} < T_{rew}^{(2)}\right] \cdot P\left(T_{rew}^{(1)} < T_{rew}^{(2)}\right) +$$
$$+ E\left[T_{rew}^{(2)} + \left(T_{rew}^{(1)} - T_{rew}^{(2)}\right) \mid T_{rew}^{(1)} > T_{rew}^{(2)}\right] \cdot P\left(T_{rew}^{(1)} > T_{rew}^{(2)}\right)$$

Since the reward times $T_{rew}^{(i)}$, $i=1,2$, are independent and exponentially distributed, we have:

$$P\left(T_{rew}^{(1)} < T_{rew}^{(2)}\right) = \frac{r_1}{r_1+r_2}$$

$$P\left(T_{rew}^{(1)} > T_{rew}^{(2)}\right) = \frac{r_2}{r_1+r_2}$$

$$E\left[T_{rew}^{(1)} + \left(T_{rew}^{(2)} - T_{rew}^{(1)}\right) \mid T_{rew}^{(1)} < T_{rew}^{(2)}\right] =$$

$$= E\left[T_{rew}^{(1)} \mid T_{rew}^{(1)} < T_{rew}^{(2)}\right] + E\left[T_{rew}^{(2)} - T_{rew}^{(1)} \mid T_{rew}^{(1)} < T_{rew}^{(2)}\right]$$

$$= \frac{1}{r_1+r_2} + \frac{1}{r_2}$$

The first part of the above expression results from the fact that the min of two independent, exp. distr. r.v.'s with rates r_1 and r_2 is an exp distr. r.v. with rate r_1+r_2.
The second part results from the memoryless property of the exp. distribution.

Working as above, we also have:

$$E\left[T_{rew}^{(2)} + T_{rew}^{(1)} - T_{rew}^{(2)} \;\middle|\; T_{rew}^{(1)} > T_{rew}^{(2)} \right] =$$

$$= \frac{1}{r_1+r_2} + \frac{1}{r_1}$$

Putting everything together, we have:

$$E\left[\max\left(T_{rew}^{(1)}, T_{rew}^{(2)}\right)\right] =$$

$$= \left(\frac{1}{r_1+r_2} + \frac{1}{r_2}\right) \cdot \frac{r_1}{r_1+r_2} + \left(\frac{1}{r_1+r_2} + \frac{1}{r_1}\right) \frac{r_2}{r_1+r_2} =$$

$$= \left(\frac{1}{0.2+0.1} + \frac{1}{0.1}\right) \frac{0.2}{0.2+0.1} + \left(\frac{1}{0.2+0.1} + \frac{1}{0.2}\right) \frac{0.1}{0.2+0.1} \text{ min} =$$

$$= \frac{0.4 \times 0.2}{0.1 \times (0.3)^2} + \frac{0.5 \times 0.1}{0.2 \times (0.3)^2} \text{ min} =$$

$$= \frac{0.4 \times 0.2^2 + 0.5 \times 0.1^2}{0.1 \times 0.2 \times (0.3)^2} \text{ min} = \frac{210}{2 \times 3^2} = \frac{105}{9} \text{ min}$$

and

$$t_e = 2 + 0.3 \times 0.8 \times \frac{1}{0.2} + 0.2 \times 0.7 \frac{1}{0.1} + 0.3 \times 0.2 \times \frac{105}{9} =$$

$$= 2 + 1.2 + 1.4 + 0.7 = 5.3 \text{ min}$$

Finally the production capacity of this line is

$$\left(\frac{1}{t_e}\right) = \frac{1}{5.3 \text{ min}} = \frac{60 \text{ min/hr}}{5.3 \text{ min}} = 11.32 \text{ hr}^{-1}$$

Remark: Strictly speaking, the line will be stable for any arrival rate that is strictly less than the above value.