Homework 2

Problem 1

Using 3.1-3.3, with $\lambda = 5/\text{hr}$, $\mu = 4/\text{hr}$

(1) $-\lambda p_0 + \mu p_1 = 0$
(2) $\lambda p_0 - (\lambda + \mu) p_1 + \mu p_2 = 0$
(3) $\lambda p_1 - (\lambda + \mu) p_2 + \mu p_3 = 0$
(4) $\lambda p_2 - (\lambda + \mu) p_3 + \mu p_4 = 0$
(5) $\lambda p_3 - \mu p_4 = 0$
and
(6) $p_0 + p_1 + p_2 + p_3 + p_4 = 1$

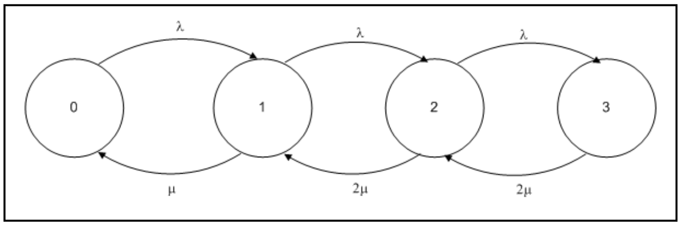
There are 6 equations and only 5 variables. Any one of the first equations (1-5) can be dropped and the system solved. Note that not using equation (6) will probably result in a solution which does not satisfy the probability postulates (that is, it won't sum to 1).

The solution (to three decimals) is:
$p_0 = 0.122,\ p_1 = 0.152,\ p_2 = 0.190,\ p_3 = 0.238,\ p_4 = 0.297$

Answers to the questions:
(a) $\lambda = \frac{120}{24} = 5$ jobs/hr
(b) Prob. of no jobs: $p_0 = 0.122$
(c) $E[N] = 0p_0 + 1p_1 + \cdots + 4p_4 = 2.437$ jobs
(d) lost jobs = $\lambda p_4 = 1.485$ jobs/hr $\times 24$ hrs = 35.64 jobs/day
(e) throughput = $\lambda_e = \lambda(1 - p_4) = 3.513$ jobs/hr
(f) CT_{sys} WIP_{sys} / λ_e 0.694 hr = 41.62 min

Problem 3

(a)



(b) isolate each single state method

0: $\lambda p_0 = \mu p_1$

1: $\lambda p_0 + 2\mu p_2 = (\lambda + \mu) p_1$

2: $\lambda p_1 + 2\mu p_3 = (\lambda + 2\mu) p_2$

3: $\lambda p_2 = 2\mu p_3$

and $\sum_{i=0}^{3} p_i = 1$.

cut between nodes: left flow = right flow across cut

0&1: $\lambda p_0 = \mu p_1$

1&2: $\lambda p_1 = 2\mu p_2$

2&3: $\lambda p_2 = 2\mu p_3$

and $\sum_{i=0}^{3} p_i = 1.$

(c) Solving using the "cut between nodes" method:

$\lambda p_0 = \mu p_1$

$\rightarrow p_1 = \frac{\lambda}{\mu} p_0$

$2\mu p_2 = \lambda p_1$

$\rightarrow p_2 = \frac{\lambda}{2\mu} p_1$

$\rightarrow p_2 = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 p_0$

$\mu p_3 = \lambda p_2$

$\rightarrow p_3 = \left(\frac{1}{2}\right)^2 \left(\frac{\lambda}{\mu}\right)^3 p_0$

$$p_0 = \frac{1}{\left(1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3\right)},$$

$$p_1 = \frac{\frac{\lambda}{\mu}}{\left(1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3\right)},$$

$$p_2 = \frac{\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2}{\left(1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3\right)},$$

$$p_3 = \frac{\frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3}{\left(1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3\right)}.$$

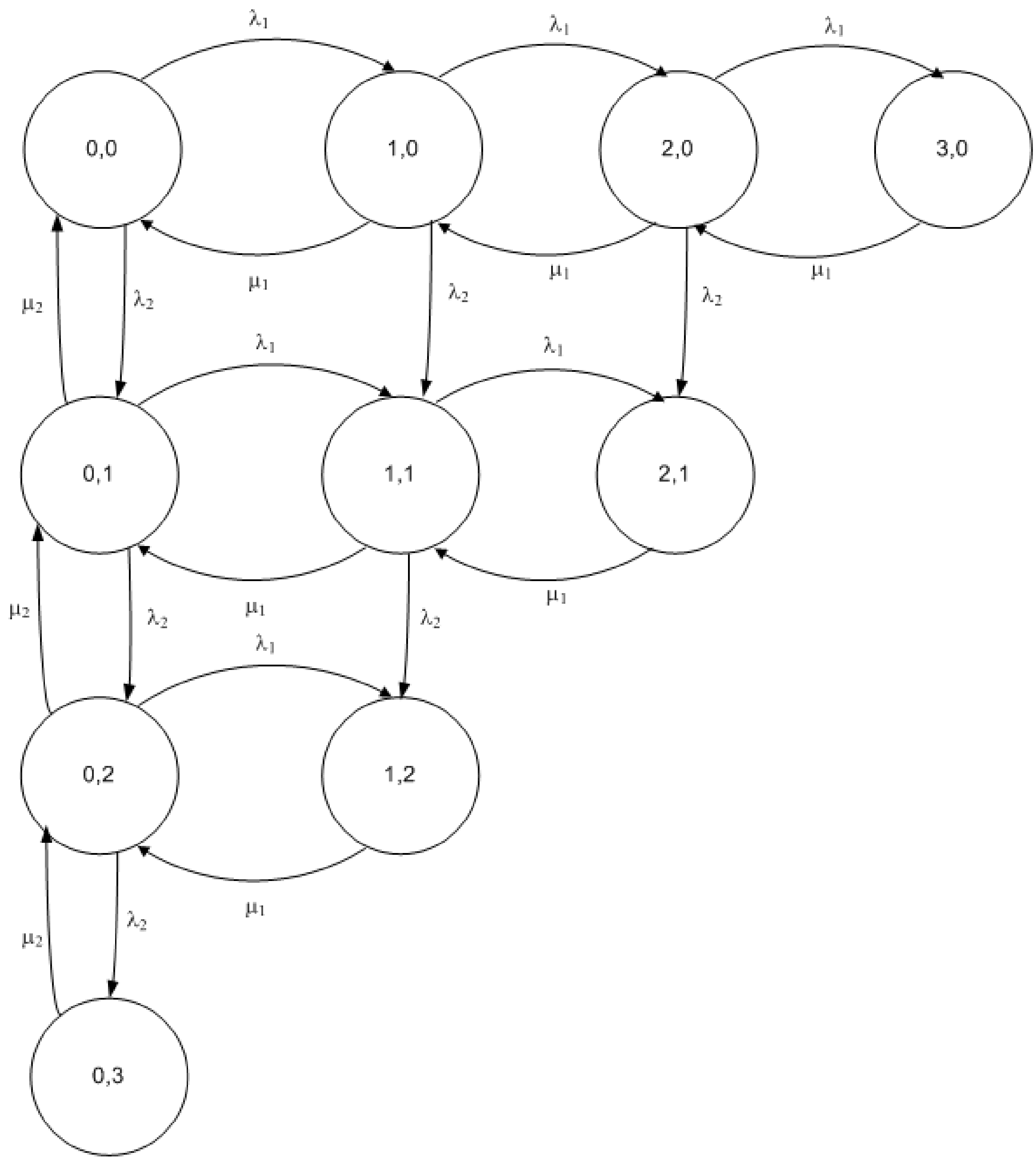
(d) Server average utilization $= 0p_0 + \frac{1}{2}p_1 + \frac{2}{2}p_2 + \frac{2}{2}p_3$.

(e) Rate of lost jobs/unit time $= \lambda p_3$.

(f) System throughput (jobs/unit time) = $\lambda_e = \lambda -$ lost jobs $= \lambda(1 - p_3)$.

Problem 4

Notation: $(i, j) = i$ jobs of type 1 and j jobs of type 2 with rates (λ_1, μ_1) and (λ_2, μ_2)



Problem 5

Using the cut between nodes method and the solution from Problem 2:

$$p_1 = \frac{\lambda}{\mu} p_0, \quad p_2 = \left(\frac{\lambda}{\mu}\right)^2 p_0, \quad p_3 = \left(\frac{\lambda}{\mu}\right)^3 p_0,$$

$$p_0\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3\right) = 1$$

and

$$p_0 = 1 \Big/ \left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3\right).$$

Throughput :

$$\lambda_e = \lambda(1 - p_3) = \lambda\left(1 - \left(\frac{\lambda}{\mu}\right)^3 p_0\right)$$

$$= \lambda \frac{\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right)}{\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3\right)}.$$

Let

$\lambda = \mu$

$$\lambda_e = \lambda\left(\frac{1+1+1}{4}\right) = \frac{3}{4}\lambda.$$

$\lambda = 2\mu$

$$\lambda_e = \lambda\left(\frac{1+2+4}{1+2+4+8}\right) = \frac{7}{15}\lambda.$$

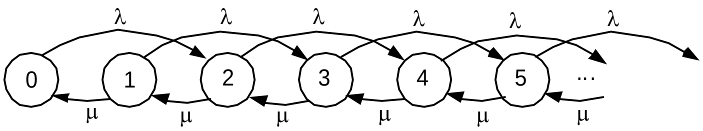
$\lambda = 3\mu$

$$\lambda_e = \lambda\left(\frac{1+3+9}{1+3+9+27}\right) = \frac{13}{40}\lambda.$$

$\lambda = 4\mu$

$$\lambda_e = \lambda\left(\frac{1+4+16}{1+4+16+64}\right) = \frac{21}{85}\lambda.$$

Problem 7



out = in

State $0: \lambda p_0 = \mu p_1$

$1: (\lambda+\mu)\, p_1 = \mu p_2$

$2: (\lambda+\mu)\, p_2 = \lambda p_0 + \mu p_3$

$3: (\lambda+\mu)\, p_3 = \lambda p_1 + \mu p_4$

$4: (\lambda+\mu)\, p_4 = \lambda p_2 + \mu p_5$

$5: (\lambda+\mu)\, p_5 = \lambda p_3 + \mu p_6$

$\vdots$

and $\sum_{i=0}^{\infty} p_i = 1$

Note that all arrivals are captured by the system. Since each arrival has two jobs and the rate of arrivals per unit time is λ, then the total arrivals per unit time is 2λ. Therefore, for the system to keep up, the service rate per unit time, μ, must be such that $2\lambda < \mu$.

Problem 16

$CT_s = CT_q + E[T_s] \rightarrow E[T_s] = CT_s - CT_q$

$CT_s = WIP_s / \lambda_e$ and $CT_q = WIP_q / \lambda_e$

$WIP_s = 0p_0 + 1p_{1f} + 1p_{1s} + 2p_2 + 3p_3 + \cdots + Np_N$

$WIP_q = 1p_3 + 2p_4 + \cdots + (N-2)p_N$

$E[T_s] = \left(1p_{1f} + 1p_{1s} + 2p_2 + 2p_3 + \cdots + 2p_N\right) / \lambda_e$

$$p_0 = \begin{cases} 1 \Big/ \left[1 + \left(\dfrac{\lambda(\lambda+\gamma)(\mu+\gamma)}{\mu\gamma(2\lambda+\gamma+\mu)}\right)\dfrac{1-r^N}{1-r}\right], & \text{if } r \neq 1 \\ 1 \Big/ \left[1 + \left(\dfrac{\lambda(\lambda+\gamma)(\mu+\gamma)}{\mu\gamma(2\lambda+\gamma+\mu)}\right)N\right], & \text{if } r = 1 \end{cases}$$

where $r = \dfrac{\lambda}{\mu+\gamma}$.

Then,

$$p_k = r^{k-1}\left(p_{1f} + p_{1s}\right),$$

and

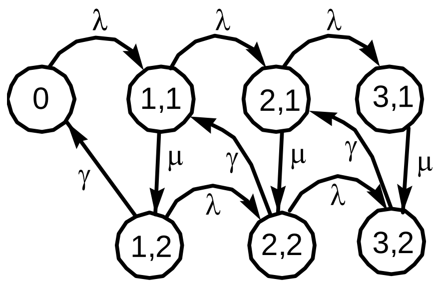
$$p_{1f} = \frac{\lambda(\lambda+\mu+\gamma)}{\mu(2\lambda+\mu+\gamma)} p_0,$$

$$p_{1s} = \frac{\lambda^2}{\gamma(2\lambda+\mu+\gamma)} p_0.$$

Problem 20

(a) States: 0, (n, m) where n is the number of jobs in system and m is the machine being worked on.

(b)



(c) Utilization server 1 $= 1p_{11} + 1p_{21} + 1p_{31}$

(d) Utilization operator $= 1p_{11} + 1p_{21} + 1p_{31} + 1p_{12} + 1p_{22} + 1p_{32}$

(e) WIP in the system: $WIP_s = 1(p_{11} + p_{12}) + 2(p_{21} + p_{22}) + 3(p_{31} + p_{32})$

(f) Throughput: $\lambda_e = \lambda(1 - p_{31} - p_{32})$