Input Data Analysis: Specifying Model Parameters & Distributions

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Overview

- Deterministic vs. random inputs
- Data collection
- Distribution fitting
  - Model “guessing”
  - Fitting parametric distributions
    - Assessment of independence
    - Parameter estimation
    - Goodness-of-fit tests
- No data?
- Non-stationary arrival processes
- Multivariate / correlated input data
- Case study
Deterministic vs. Random Inputs

- *Deterministic*: Nonrandom, fixed values
  - Number of units of a resource
  - Entity transfer time (?)
  - Interarrival, processing times (?)

- *Random*: Model as a distribution, “draw” or “generate” values from to drive simulation
  - Interarrival, processing times
  - What distribution? What distributional parameters?
  - Causes simulation output to be random, too

- Don’t just assume randomness away!
Collecting Data

- Generally hard, expensive, frustrating, boring
  - System might not exist
  - Data available on the wrong things — might have to change model according to what’s available
  - Incomplete, “dirty” data
  - Too much data (!)
- Sensitivity of outputs to uncertainty in inputs
- Match model detail to quality of data
- Cost — should be budgeted in project
- Capture variability in data — model validity
- Garbage In, Garbage Out (GIGO)
Using Data: Alternatives and Issues

- Use data “directly” in simulation
  - Read actual observed values to drive the model inputs (interarrivals, service times, part types, ...)
  - All values will be “legal” and realistic
  - But can never go outside your observed data
  - May not have enough data for long or many runs
  - Computationally slow (reading disk files)

- Or, fit probability distribution to data
  - “Draw” or “generate” synthetic observations from this distribution to drive the model inputs
  - Can go beyond observed data (good and bad)
  - May not get a good “fit” to data — validity?
Fitting Distributions: Some Important Issues

- Not an exact science — no “right” answer
- Consider theoretical vs. empirical
- Consider range of distribution
  - Infinite both ways (e.g., normal)
  - Positive (e.g., exponential, gamma)
  - Bounded (e.g., beta, uniform)
- Consider ease of parameter manipulation to affect means, variances
- Simulation model sensitivity analysis
- Outliers, multimodal data
  - Maybe split data set
Guess model using:

- Summary statistics, such as
  - Sample mean $\overline{X}_n$
  - Sample variance $S_n^2$
  - Sample median
  - Sample coefficient of variation $S_n/\overline{X}_n$
  - Sample skewness

- Skewness close to zero indicates a symmetric distribution
- A skewed distribution with unit coefficient of variation is likely the exponential

- Histograms (play with interval width to get a reasonably smooth histogram). They resemble the unknown density
- Box plots

Estimates

$CV(X) = \frac{\sigma}{\mu} = \sqrt{\frac{\text{Var}(X)}{\text{E}(X)}}$

$\frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X}_n)^3$

$E(X - \mu)^3 / \sigma^3$

$\frac{S_n^3}{n}$
Main Steps (continued)

- If a parametric model seems plausible:
  - Estimate parameters
  - Test goodness-of-fit
Fitting Parametric Distributions

- Assume that the sample data are independent identically distributed data from some distribution with density (probability) function

\[ X_1, X_2, \ldots, X_n \sim f(x; \theta) \]

\[ \theta = (\theta_1, \ldots, \theta_m) \]

- All data are complete (no censoring)
- How can we test independence?
  - Using the scatter-plot of \((X_i, X_{i+1}), i = 1, \ldots, n - 1\)
  - By means of von-Neumann’s test
Von Neumann’s Test

The test statistic is

\[ U_n = \sqrt{\frac{n^2 - 1}{n - 2}} \times \left[ \hat{\rho}_1 + \frac{(X_1 - \bar{X}_n)^2 + (X_n - \bar{X}_n)^2}{2 \sum_{i=1}^{n} (X_i - \bar{X}_n)^2} \right] \]

where

\[ \hat{\rho}_1 = \frac{\sum_{i=1}^{n-1} (X_i - \bar{X}_n)(X_{i+1} - \bar{X}_n)}{\sum_{i=1}^{n} (X_i - \bar{X}_n)^2} \]

estimates the correlation between adjacent observations.

If the data are independent and \( n \geq 20 \), \( U_n \approx N(0, 1) \)

We reject the hypothesis of independence when

\[ |U_n| > z_{\beta/2} \]

where \( \beta \) is the type-I error.
Types of Parameters

- **Location** parameters — they shift the density function
- **Shape** parameters — they change the shape of the density function
- **Scale** parameters

**Example:** For the \( N(\mu, \sigma^2) \) distribution
- \( \mu \) is the location parameter because \( X \sim N(\mu, \sigma^2) \Leftrightarrow X-\mu \sim N(0, \sigma^2) \)
- \( \sigma \) is the scale parameter because \( X \sim N(\mu, \sigma^2) \Leftrightarrow X/\sigma \sim N(\mu, 1) \)

**Example:** In the Weibull(\( \alpha, \lambda \)) distribution
- \( \alpha \) is a shape parameter
- \( \lambda \) is the scale parameter
Parameter Estimation Methods

- Method of moments
- Maximum likelihood estimation
Method of Moments

- Equate the first $m$ sample (non-central) moments to the theoretical moments and solve the resulting system for the unknown parameters:

$$E(X^k) = \frac{1}{n} \sum_{i=1}^{n} X_i^k, \; k = 1, \ldots, m$$
Example: The normal distribution

\[ E(X) = \mu = \bar{X}_n \]

\[ E(X^2) = \mu^2 + \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2 \]

give

\[ \hat{\mu} = \bar{X}_n \quad \text{and} \quad \hat{\sigma} = S_n \]
Maximum Likelihood Estimation

- The likelihood function is the joint density (probability function) of the data:

\[ L(\theta) = \prod_{i=1}^{n} f(X_i; \theta) \]

- The Maximum Likelihood Estimator of \( \theta \) maximizes \( L(\theta) \) or, equivalently, the log-likelihood \( \ln L(\theta) \):

\[ \ln L(\hat{\theta}) \geq \ln L(\theta) \text{ for all } \theta \]
Example: The exponential distribution

\[
\ell(\lambda) \equiv \ln L(\lambda) = \ln \left( \prod_{i=1}^{n} \lambda e^{-\lambda X_i} \right) = n \ln \lambda - \lambda \sum_{i=1}^{n} X_i
\]

\[
\frac{d\ell}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0 \Rightarrow \hat{\lambda} = 1/\bar{X}_n
\]

Check that \(d^2\ell / d\lambda^2 = -1/\lambda^2 < 0\); this guarantees that \(\hat{\lambda}\) is a maximizer
Example: The normal distribution

\[ \hat{\mu} = \bar{X}_n \]

\[ \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 = \frac{n-1}{n} S_n^2 \]
Example: The Uniform(0, \(b\)) distribution

We wish to find the MLE of \(b\)

The likelihood function is

\[
L(b) = \begin{cases} 
1 / b^n & \text{for } 0 \leq X_i \leq b \iff b \geq \max X_i \\
0 & \text{otherwise}
\end{cases}
\]

Notice that \(L(b)\) is discontinuous; so don’t take derivatives...

Check that \(L(b)\) is maximized at

\[
\hat{b} = \max X_i
\]
Example: The Weibull distribution

The density is given by

\[ f(x) = \alpha \lambda (\lambda x)^{\alpha-1} \exp[-(\lambda x)^\alpha], \]

where \( \alpha > 0 \) is the shape parameter and \( \lambda > 0 \) is the scale parameter.

The m.l.e.s satisfy the following equations:

\[
\frac{\sum_{i=1}^{n} X_i^\hat{\alpha} \ln X_i}{\sum_{i=1}^{n} X_i^\hat{\alpha}} - \frac{1}{\hat{\alpha}} = \frac{\sum_{i=1}^{n} \ln X_i}{n} \quad \text{and} \quad \hat{\lambda} = \left( \frac{\sum_{i=1}^{n} X_i^\hat{\alpha}}{n} \right)^{-1/\hat{\alpha}}
\]

We can solve the first equation by Newton’s method.
MLEs are “nice” because they are
- Asymptotically \((n \to \infty)\) unbiased
- Asymptotically normal
- Invariant, i.e., if \(g\) is continuous,

\[
\lambda = g(\theta) \Rightarrow \hat{\lambda} = g(\hat{\theta})
\]

**Example:** The MLE of the variance \((\sigma^2 = 1/\lambda^2)\) for the exponential distribution is \(\overline{X}_n^2\)
Testing Goodness-of-Fit

We want to test the null hypothesis

\[ H_0 : X_1, \ldots, X_n \text{ are from } \hat{f}(x) = f(x; \hat{\theta}) \]

\( \alpha = \text{Type I Error} = \Pr(\text{reject } H_0 | H_0 \text{ is true}) \)
\( \beta = \text{Type II Error} = \Pr(\text{accept } H_0 | H_0 \text{ is false}) \)
\( \text{Power} = 1 - \beta = \Pr(\text{reject } H_0 | H_0 \text{ is false}) \)
\( p\text{-value} = \text{smallest value of type I error that leads to rejection of } H_0 \)
Graphical approaches

The Q-Q plot graphs the quantiles of the fitted distribution vs. the sample quantiles. It emphasizes poor fitting at the tails.

The P-P plot graphs the fitted CDF vs. the empirical CDF.

\[
\bar{F}(x) = \frac{\text{number of } X_i \leq x}{n}, \quad -\infty < x < \infty
\]

Computation: Sort \( X_{(1)} < X_{(2)} < \cdots < X_{(n)} \). Then

\[
\bar{F}(X_{(i)}) = \frac{i}{n}
\]

It emphasizes poor fitting at the middle of the fitted CDF.
Testing Goodness-of-Fit (continued)

- Statistical Tests
  - The chi-square test
  - The Kolmogorov-Smirnov test
  - The Anderson-Darling test
The Chi-square Test

- Split the range of $X$ into $k$ adjacent intervals

- Let

  $$I_i = [a_{i-1}, a_i) = \text{ith interval}$$

  $$O_i = \text{number of observations in interval } i$$

  $$E_i = \text{expected number of observations in interval } i = n[\hat{F}(a_i) - \hat{F}(a_{i-1})]$$

  CDF of fitted distribution
The Chi-square Test (continued)

- The null hypothesis is rejected (at level \( \alpha \)) if

\[
\chi_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} > \chi_{k-s-1,\alpha}^2
\]

where \( s \) is the number of parameters replaced by their MLEs

- One should use \( E_i \geq 5 \)
- The test has maximum power if the \( E_i \) are equal (the intervals are equiprobable)
The Kolmogorov-Smirnov Test

- It generally assumes that all parameters are known
- Sort the data and define the empirical CDF

\[
\bar{F}(x) = \frac{\text{number of } X_i \leq x}{n} = \begin{cases} 
0 & \text{if } x < X_{(1)} \\
\frac{i}{n} & \text{if } X_{(i)} \leq x < X_{(i+1)}, \ 1 \leq i \leq n - 1 \\
1 & \text{if } x > X_{(n)}
\end{cases}
\]
The null hypothesis is rejected (at level $\alpha$) if

$$D_n = \sup \left| \hat{F}(x) - \bar{F}(x) \right|$$

$$= \max \left\{ \max \left[ \frac{i}{n} - \hat{F}(X_{(i)}) \right], \max \left[ \hat{F}(X_{(i)}) - \frac{i - 1}{n} \right] \right\} > d_{n, \alpha}$$
The Kolmogorov-Smirnov Test (continued)

- We usually simplify the above inequality by computing a modified test statistic and a modified critical value $c_\alpha$:

  \[ \text{Adjusted Test Statistic} > \underbrace{c_\alpha}_{\text{tabulated}} \]

- When parameters are replaced by MLEs modified K-S test statistics exist for the following distributions:
  - Normal
  - Exponential
  - Weibull
  - Log-logistic
The Kolmogorov-Smirnov Test (continued)

Modified Critical Values $c_{\alpha}$ for Adjusted K-S Statistics

<table>
<thead>
<tr>
<th>Case</th>
<th>Adjusted Test Statistic</th>
<th>(0.15)</th>
<th>(0.10)</th>
<th>(0.05)</th>
<th>(0.025)</th>
<th>(0.01)</th>
</tr>
</thead>
<tbody>
<tr>
<td>All parameters</td>
<td>(\left(\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}}\right) D_n)</td>
<td>1.138</td>
<td>1.224</td>
<td>1.358</td>
<td>1.480</td>
<td>1.628</td>
</tr>
<tr>
<td>known</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nor((\bar{X}_n, S_n^2))</td>
<td>(\left(\sqrt{n} - 0.01 + \frac{0.85}{\sqrt{n}}\right) D_n)</td>
<td>0.775</td>
<td>0.819</td>
<td>0.895</td>
<td>0.995</td>
<td>1.035</td>
</tr>
<tr>
<td>Expo((1/\bar{X}_n))</td>
<td>(\left(D_n - \frac{0.2}{\sqrt{n}}\right) \left(\sqrt{n} - 0.01 + \frac{0.85}{\sqrt{n}}\right))</td>
<td>0.926</td>
<td>0.990</td>
<td>1.094</td>
<td>1.190</td>
<td>1.308</td>
</tr>
</tbody>
</table>
The Kolmogorov-Smirnov Test (continued)

Modified Critical Values for the K-S Test for the Weibull Distribution

<table>
<thead>
<tr>
<th>n</th>
<th>( \alpha )</th>
<th>( \alpha = 0.10 )</th>
<th>( \alpha = 0.05 )</th>
<th>( \alpha = 0.025 )</th>
<th>( \alpha = 0.01 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.760</td>
<td>0.819</td>
<td>0.880</td>
<td>0.944</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.779</td>
<td>0.843</td>
<td>0.907</td>
<td>0.973</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.790</td>
<td>0.856</td>
<td>0.922</td>
<td>0.988</td>
<td></td>
</tr>
<tr>
<td>( \infty )</td>
<td>0.803</td>
<td>0.874</td>
<td>0.939</td>
<td>1.007</td>
<td></td>
</tr>
</tbody>
</table>
The Kolmogorov-Smirnov Test (continued)

Modified Critical Values for the K-S Test for the Log-logistic Distribution

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.10</td>
</tr>
<tr>
<td>10</td>
<td>0.679</td>
</tr>
<tr>
<td>20</td>
<td>0.698</td>
</tr>
<tr>
<td>50</td>
<td>0.708</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.715</td>
</tr>
</tbody>
</table>
The Anderson-Darling Test

The null hypothesis is rejected (at level $\alpha$) if

$$A_n^2 = n \int_{-\infty}^{\infty} \frac{[\hat{F}(x) - \bar{F}(x)]^2}{\hat{F}(x)[1 - \hat{F}(x)]} f(x) \, dx$$

$$= -\frac{1}{n} \sum_{i=1}^{n} (2i - 1) \left\{ \ln \hat{F}(X_{(i)}) + \ln\left[1 - \hat{F}(X_{(n-i+1)})\right] \right\} - n > a_{n,1-\alpha} \text{ (tabulated)}$$

It generally assumes that all parameters are known.
The Anderson-Darling Test (continued)

- We usually simplify the above inequality by computing a modified test statistic and a modified critical value $a_\alpha$:

  **Adjusted Test Statistic** > $a_\alpha$ tabulated

- When parameters are replaced by MLEs, modified A-D test statistics exist for:
  - The normal distribution
  - The exponential distribution
  - The Weibull distribution
  - The log-logistic distribution
## The Anderson-Darling Test (continued)

Modified Critical Values $a_\alpha$ for Adjusted A-D Statistics

<table>
<thead>
<tr>
<th>Case</th>
<th>Adjusted Test Statistic</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0.10</td>
</tr>
<tr>
<td>All parameters known</td>
<td>$A_n^2$ for $n \geq 5$</td>
<td>1.933</td>
</tr>
<tr>
<td>Nor($\bar{X}_n, S^2_n$)</td>
<td>$(1 + \frac{4}{n} - \frac{25}{n^2}) A_n^2$</td>
<td>0.632</td>
</tr>
<tr>
<td>Expo($1/\bar{X}_n$)</td>
<td>$(1 + \frac{0.6}{n}) A_n^2$</td>
<td>1.070</td>
</tr>
<tr>
<td>Weibull($\hat{\alpha}, \hat{\beta}$)</td>
<td>$(1 + \frac{0.2}{\sqrt{n}}) A_n^2$</td>
<td>0.637</td>
</tr>
<tr>
<td>Log-logistic($\hat{\alpha}, \hat{\beta}$)</td>
<td>$(1 + \frac{0.25}{\sqrt{n}}) A_n^2$</td>
<td>0.563</td>
</tr>
</tbody>
</table>
No Data?

- Happens more often than you would like
- No good solution; some (bad) options:
  - Interview “experts”
    - Min, Max: Uniform
    - Average, % error or absolute error: Uniform
    - Min, Mode, Max: Triangular
      - Mode can be different from Mean — allows asymmetry (skewness)
  - Interarrivals — independent, stationary
    - Exponential — still need some value for mean
  - Number of “random” events in an interval: Poisson
  - Sum of independent “pieces”: normal
  - Product of independent “pieces”: lognormal
Non-stationary Arrival Processes

- External events (often arrivals) whose rate varies over time
  - Lunchtime at fast-food restaurants
  - Rush-hour traffic in cities
  - Telephone call centers
  - Seasonal demands for a manufactured product

- It can be critical to model this non-stationarity for model validity
  - Ignoring peaks, valleys can mask important behavior
  - Can miss rush hours, etc.

- Good model: *Non-stationary Poisson process*
Non-stationary Arrival Processes (continued)

- Two issues:
  - How to specify/estimate the rate function
  - How to generate from it properly during the simulation (will be discussed during the Output Analysis session)

- Several ways to estimate rate function — we’ll just do the piecewise-constant method
  - Divide time frame of simulation into subintervals of time over which you think rate is fairly flat
  - Compute observed rate within each subinterval
  - Be very careful about time units!
    - Model time units = minutes
    - Subintervals = half hour (= 30 minutes)
    - 45 arrivals in the half hour; rate = 45/30 = 1.5 per minute
Multivariate and Correlated Input Data

- Usually we assume that all generated random observations across a simulation are independent (though from possibly different distributions)

- Sometimes this isn’t true:
  - A “difficult” part may require longer service times by a set of machines
  - This indicates positive correlation

- Ignoring such relations can invalidate model
Case Study: Times-to-Failure

- A data set contains 200 times-to-failure for a piece of equipment
- We use ExpertFit®
- To assess independence, we create a scatter plot
Case Study — Scatter Plot

The data appear to be independent
Case Study — Data Summary

<table>
<thead>
<tr>
<th>Data Characteristic</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Source file</td>
<td>TTF.DAT</td>
</tr>
<tr>
<td>Observation type</td>
<td>Real valued</td>
</tr>
<tr>
<td>Number of observations</td>
<td>200</td>
</tr>
<tr>
<td>Minimum observation</td>
<td>162.26205</td>
</tr>
<tr>
<td>Maximum observation</td>
<td>2,351.98858</td>
</tr>
<tr>
<td>Mean</td>
<td>768.91946</td>
</tr>
<tr>
<td>Median</td>
<td>709.90162</td>
</tr>
<tr>
<td>Variance</td>
<td>157,424.22579</td>
</tr>
<tr>
<td>Coefficient of variation</td>
<td>0.51601</td>
</tr>
<tr>
<td>Skewness</td>
<td>1.02670</td>
</tr>
</tbody>
</table>

- Can the data be from
  - The normal distribution?
  - The exponential distribution?
Case Study — Histogram with 16 Intervals

<table>
<thead>
<tr>
<th>Interval Midpoint</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.3e2</td>
<td>5.2e2</td>
</tr>
<tr>
<td>5.2e2</td>
<td>8.0e2</td>
</tr>
<tr>
<td>8.0e2</td>
<td>10.9e2</td>
</tr>
<tr>
<td>10.9e2</td>
<td>13.8e2</td>
</tr>
<tr>
<td>13.8e2</td>
<td>16.6e2</td>
</tr>
<tr>
<td>16.6e2</td>
<td>19.5e2</td>
</tr>
<tr>
<td>19.5e2</td>
<td>22.4e2</td>
</tr>
</tbody>
</table>

16 intervals of width 143.325 between 160 and 2,453.2
Case Study — Model Guessing

- We will allow ExpertFit to choose a continuous distribution automatically.
- We will tell it that
  - the left limit for the underlying random variable is zero and
  - the tight limit is infinity.
Case Study — ExpertFit’s Choice...

### Relative Evaluation of Candidate Models

<table>
<thead>
<tr>
<th>Model</th>
<th>Relative Score</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - Weibull(E)</td>
<td>100.00</td>
<td>Location: 161.74177</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Scale: 673.46506</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Shape: 1.54741</td>
</tr>
<tr>
<td>2 - Beta</td>
<td>95.45</td>
<td>Lower endpoint: 54.43617</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Upper endpoint: 12,916.87962</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Shape #1: 3.00707</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Shape #2: 51.12749</td>
</tr>
<tr>
<td>3 - Gamma</td>
<td>89.77</td>
<td>Location: 0.00000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Scale: 197.09191</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Shape: 3.90132</td>
</tr>
</tbody>
</table>

23 models are defined with scores between 0.00 and 100.00

### Absolute Evaluation of Model 1 - Weibull(E)

Evaluation: Good
Suggestion: Additional evaluations using Comparisons Tab might be informative.

### Additional Information About Model 1 - Weibull(E)

- Results of the Anderson-Darling goodness-of-fit test at level 0.1: Not applicable
- "Error" in the model mean relative to the sample mean: 1.35980 - 0.18%

**Weibull(E): Weibull distribution with a location parameter**
Case Study — Histogram Comparisons

The gamma distribution does not fit well at the left tail...
Case Study — Graphical Goodness-of-Fit Tests

P-P Plot

- Range of sample
- 1 - Weibull(E) (discrepancy=0.02285)
- 3 - Gamma (discrepancy=0.03057)
Case Study — Graphical Goodness-of-Fit Tests

(continued)

![Q-Q Plot](image-url)

1 - Weibull(E) (discrepancy=0.05400)
3 - Gamma (discrepancy=0.05316)
# Case Study — A-D & K-S Goodness-of-Fit Tests

## Anderson-Darling Test With Model 1 - Weibull(E)

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>Sample Size Critical Values for Level of Significance (alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>1.248 1.933 2.492 3.070 3.857 4.500</td>
</tr>
</tbody>
</table>

### Note:
- No critical values exist for this special case.
- The following critical values are for the case where all parameters are known, and are conservative.

## Kolmogorov-Smirnov Test With Model 1 - Weibull(E)

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>Sample Size Critical Values for Level of Significance (alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>1.128 1.213 1.346 1.467 1.613</td>
</tr>
</tbody>
</table>

### Note:
- No critical values exist for this special case.
- The following critical values are for the case where all parameters are known, and are conservative.

## Anderson-Darling Test With Model 3 - Gamma

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>Sample Size Critical Values for Level of Significance (alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>0.474 0.638 0.761 0.884 1.047 1.176</td>
</tr>
</tbody>
</table>

### Note:
- The following critical values are approximate.

## Kolmogorov-Smirnov Test With Model 3 - Gamma

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>Sample Size Critical Values for Level of Significance (alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>1.128 1.213 1.346 1.467 1.613</td>
</tr>
</tbody>
</table>

### Note:
- No critical values exist for this special case.
- The following critical values are for the case where all parameters are known, and are conservative.
Case Study — Chi-square Goodness-of-Fit Tests

Equal-Probable Chi-Square Test With Model 1 - Weibull(E)

<table>
<thead>
<tr>
<th>Degrees of Freedom</th>
<th>Observed Level of Significance</th>
<th>Critical Values for Level of Significance (alpha)</th>
<th>0.25</th>
<th>0.15</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>0.554</td>
<td>19.369</td>
<td>21.793</td>
<td>23.542</td>
<td>25.296</td>
<td>26.700</td>
<td>32.000</td>
</tr>
<tr>
<td>19</td>
<td>0.748</td>
<td>22.718</td>
<td>25.329</td>
<td>27.204</td>
<td>30.144</td>
<td>36.191</td>
<td></td>
</tr>
<tr>
<td>Reject?</td>
<td>No</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Warning: The test may not be statistically valid because a method other than maximum likelihood was used to estimate parameters.

Beware: Outcomes depend on the number of intervals!

Equal-Probable Chi-Square Test With Model 3 - Gamma

<table>
<thead>
<tr>
<th>Degrees of Freedom</th>
<th>Observed Level of Significance</th>
<th>Critical Values for Level of Significance (alpha)</th>
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<th>0.15</th>
<th>0.10</th>
<th>0.05</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>0.045</td>
<td>20.489</td>
<td>22.977</td>
<td>24.769</td>
<td>27.587</td>
<td>33.409</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>0.083</td>
<td>22.718</td>
<td>25.329</td>
<td>27.204</td>
<td>30.144</td>
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<td></td>
<td>No</td>
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</tbody>
</table>

What distribution gives a better fit?
Case Study — Additional Graphical Comparisons

Box-Plot Comparisons

- 7-point sample box plot
- 1 - Weibull(E)
- 3 - Gamma
Case Study — Arena Code for the Winner...

Arena Representation of Model 1 - Weibull(E)

Use:

\[ 161.741769 + \text{WEIB}(673.465060, 1.547408, \text{<stream>}) \]

1. **Estimate for location parameter.** Check the translation...
2. **Estimate for scale parameter**
3. **Estimate for shape parameter**

We haven't used this yet...