

ISyE 6761 — Fall 2012

Homework #5 Solutions (Revised 12/9/12)

1. For a renewal process, let $A(t)$ be the age at time t . Prove that if the interarrival mean $\mu < \infty$, then w.p.1, $A(t)/t \rightarrow 0$ as $t \rightarrow \infty$.

Solution: Note that

$$\frac{A(t)}{t} = \frac{t - S_{N(t)}}{t} = 1 - \frac{S_{N(t)}}{t} = 1 - \frac{S_{N(t)}}{N(t)} \frac{N(t)}{t}.$$

By the SLLN, $S_{N(t)}/N(t) \rightarrow \mu$. Then since $N(t)/t \rightarrow 1/\mu$, we're done. \square

2. Suppose $A(t)$ and $B(t)$ are the age and remaining life of a renewal process having interarrival c.d.f. F , find an expression for $\mathbf{P}(B(t) > x | A(t) = s)$.

Solution: We have

$$\begin{aligned} \mathbf{P}(B(t) > x | A(t) = s) &= \mathbf{P}(\text{no renewals in } (t, t+x] | A(t) = s) \\ &= \mathbf{P}(\text{interarrival} > x + s, | A(t) = s) \\ &= \mathbf{P}(\text{interarrival} > x + s, | \text{interarrival} > s) \\ &= \frac{1 - F(x + s)}{1 - F(s)}. \quad \square \end{aligned}$$

3. Find an expression for the long-run proportion of time that $a < X_{N(t)+1} < b$. Simplify if the interarrival distribution $X \sim \text{Exp}(\lambda)$. Simplify if possible for $X \sim \text{Erlang}_k(\lambda)$.

Solution: There are at least a couple of ways to do this problem. Here's a straightforward (but maybe slightly tedious) method.

First of all, note that the joint c.d.f. of two RV's X and Y can be written as (draw a picture if you don't believe me)

$$\mathbf{P}(X \leq x, Y \leq y) = \mathbf{P}(X > x, Y > y) + \mathbf{P}(X \leq x) + \mathbf{P}(Y \leq y) - 1.$$

Thus, the joint p.d.f. (assuming X and Y are continuous) is

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} \mathbf{P}(X \leq x, Y \leq y) = \frac{\partial^2}{\partial x \partial y} \mathbf{P}(X > x, Y > y).$$

In particular, let $f_{A(t),B(t)}(x, y)$ denote the p.d.f. of the age $A(t)$ and remaining life $B(t)$ of a renewal process. Then

$$\begin{aligned} f_{A(t),B(t)}(x, y) &= \frac{\partial^2}{\partial x \partial y} \mathbf{P}(A(t) > x, B(t) > y) \\ &\rightarrow \frac{1}{\mu} \frac{\partial^2}{\partial x \partial y} \int_{x+y}^{\infty} (1 - F(u)) du, \quad \text{for large } t \end{aligned}$$

where the last result follows from class notes, and where μ and $F(t)$ are the mean and c.d.f. of the interarrival times.

Continuing, we have

$$\begin{aligned} f_{A(t),B(t)}(x, y) &\rightarrow \frac{1}{\mu} \frac{\partial^2}{\partial x \partial y} \int_x^{\infty} (1 - F(s + y)) ds \\ &= \frac{1}{\mu} \frac{\partial}{\partial y} [F(x + y) - 1] \\ &= \frac{f(x + y)}{\mu}, \end{aligned}$$

where $f(\cdot)$ is the p.d.f. of the interarrival times.

Recalling from class notes that $X_{N(t)+1} = C(t) = A(t) + B(t)$, we have

$$\begin{aligned} \mathbf{P}(X_{N(t)+1} < z) &= \int \int_{x+y < z} f_{A(t),B(t)}(x, y) dx dy \\ &\rightarrow \int_0^z \int_0^{z-x} \frac{f(x+y)}{\mu} dy dx \\ &= \frac{1}{\mu} \int_0^z [F(z) - F(x)] dx \\ &= \frac{1}{\mu} \left[zF(z) - \int_0^z F(x) dx \right] \\ &= \frac{1}{\mu} \left[zF(z) - F(x)x \Big|_0^z + \int_0^z xf(x) dx \right] \quad (\text{by parts}) \\ &= \frac{1}{\mu} \int_0^z xf(x) dx. \end{aligned}$$

Thus, $\mathbf{P}(a < X_{N(t)+1} < b) \rightarrow \frac{1}{\mu} \int_a^b xf(x) dx$ for large t . \square

Here is another way to solve the problem. The long-run proportion of time that $X_{N(t)+1} \in (a, b)$ is by definition

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \frac{\sum_{i=1}^{N(t)} X_i \mathbb{1}[a < X_i < b] + B(t) \mathbb{1}[a < X_{N(t)+1} < b]}{\sum_{i=1}^{N(t)} X_i + B(t)} \\
& \leq \lim_{t \rightarrow \infty} \frac{\sum_{i=1}^{N(t)} X_i \mathbb{1}[a < X_i < b] + B(t)}{\sum_{i=1}^{N(t)} X_i + B(t)} \\
& = \lim_{t \rightarrow \infty} \frac{\left[\sum_{i=1}^{N(t)} X_i \mathbb{1}[a < X_i < b] + B(t) \right] / N(t)}{\left[\sum_{i=1}^{N(t)} X_i + B(t) \right] / N(t)} \\
& = \frac{\mathbb{E} \left[X_i \mathbb{1}[a < X_i < b] \right]}{\mathbb{E}[X_i]} \quad (\text{SLLN, and } \lim_{t \rightarrow \infty} \frac{B(t)}{N(t)} = 0 \text{ if } \mathbb{E}[X_i] < \infty) \\
& = \frac{1}{\mu} \int_a^b x f(x) dx.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \frac{\sum_{i=1}^{N(t)} X_i \mathbb{1}[a < X_i < b] + B(t) \mathbb{1}[a < X_{N(t)+1} < b]}{\sum_{i=1}^{N(t)} X_i + B(t)} \\
& \geq \lim_{t \rightarrow \infty} \frac{\sum_{i=1}^{N(t)} X_i \mathbb{1}[a < X_i < b]}{\sum_{i=1}^{N(t)} X_i + B(t)} \\
& = \lim_{t \rightarrow \infty} \frac{\left[\sum_{i=1}^{N(t)} X_i \mathbb{1}[a < X_i < b] \right] / N(t)}{\left[\sum_{i=1}^{N(t)} X_i + B(t) \right] / N(t)} \\
& = \frac{\mathbb{E} \left[X_i \mathbb{1}[a < X_i < b] \right]}{\mathbb{E}[X_i]} \quad (\text{SLLN}) \\
& = \frac{1}{\mu} \int_a^b x f(x) dx.
\end{aligned}$$

By squeezing, we have

$$\lim_{t \rightarrow \infty} \frac{\sum_{i=1}^{N(t)} X_i \mathbb{1}[a < X_i < b] + B(t) \mathbb{1}[a < X_{N(t)+1} < b]}{\sum_{i=1}^{N(t)} X_i + B(t)} = \frac{1}{\mu} \int_a^b x f(x) dx. \quad \square$$

If $X \sim \text{Exp}(\lambda)$, we have

$$\begin{aligned}
 \mathbb{P}(a < X_{N(t)+1} < b) &\rightarrow \frac{1}{\mu} \int_a^b x \lambda e^{-\lambda x} dx \\
 &= \lambda \int_a^b x e^{-\lambda x} dx \\
 &= \lambda \left[-x e^{-\lambda x} \Big|_a^b + \int_a^b e^{-\lambda x} dx \right] \\
 &= (\lambda a + 1)e^{-\lambda a} - (\lambda b + 1)e^{-\lambda b}. \quad \square
 \end{aligned}$$

Note that if $a = 0$, this simplifies to the Erlang₂(λ) c.d.f., which makes sense since $A(t)$ and $B(t)$ are i.i.d. $\text{Exp}(\lambda)$ (as we know from class notes).

Finally, we do the exercise when $f(x)$ is itself Erlang₂(λ). In this case,

$$\begin{aligned}
 \mathbb{P}(X_{N(t)+1} < z) &= \frac{1}{\mu} \int_a^b x f(x) dx \\
 &= \frac{\lambda}{2} \int_a^b x \lambda^2 x e^{-\lambda x} dx \\
 &= F_3(b) - F_3(a) \quad (\text{after some algebra}),
 \end{aligned}$$

where $F_3(x) = 1 - e^{-\lambda x}(\frac{\lambda^2 x^2}{2} + \lambda x + 1)$ is the Erlang₃(λ) c.d.f. \square

4. Customers arrive at a single-server system according to a $\text{PP}(\lambda)$. Upon arriving, a customer must pass through a door that leads to the server. However, each time someone passes through, the door becomes locked for the next t units of time. An arrival finding a locked door is lost, and a cost c is incurred by the system (perhaps the customer gets mad and causes damage). An arrival finding the door unlocked goes through to the server. If the server is free, the customer gets served; if the server is busy, the customer leaves without getting served, and a cost of K is incurred. The service time of a customer $\text{Exp}(\mu)$.

- (a) Find the average cost per unit time incurred by the system.

Solution: (almost verbatim from Baby Ross, Example 7.14) This is a renewal-reward process with a new cycle beginning each time a customer arrives to find the door unlocked. (This follows since whether or not the arrival finds the server free, the door will lock for the next t time units and the server will

be busy for a time $X \sim \text{Exp}(\mu)$. If the server is free, X is the service time of that entering customer; if the server is busy, X is the remaining time of the customer in service.) Since the next cycle will begin at the first arrival after a time t has passed, we have

$$\mathbb{E}[\text{time of cycle}] = t + \frac{1}{\lambda}.$$

Let C_1 be the cost incurred during a cycle due to arrivals finding a locked door. Then since each arrival in the first t time units of a cycle will result in a cost c , we have $\mathbb{E}[C_1] = \lambda tc$.

In addition, let C_2 be the cost incurred in a cycle due to an arrival finding the door unlocked but the server busy. Then because a cost K is incurred if the server is still busy a time t after the cycle began and, in addition, the next arrival after that time occurs before the service completion, we have

$$\mathbb{E}[C_2] = Ke^{-\mu t} \frac{\lambda}{\lambda + \mu}.$$

Thus, the average cost per unit time is

$$\frac{\lambda tc + \frac{\lambda Ke^{-\mu t}}{\lambda + \mu}}{t + \frac{1}{\lambda}}. \quad \square$$

- (b) Find the long-run proportion of time that the server is busy.

Solution: Let B be the amount of time that the server is busy in a cycle. Let Y be the remaining service time of the person in service at the beginning of the cycle. Then

$$\begin{aligned} \mathbb{E}[B] &= \mathbb{E}[B|Y < t]\mathbb{P}(Y < t) + \mathbb{E}[B|Y > t]\mathbb{P}(Y > t) \\ &= \mathbb{E}[Y|Y < t]\mathbb{P}(Y < t) + \left(t + \frac{1}{\lambda + \mu}\right)e^{-\mu t} \\ &= \int_0^t y\mu e^{-\mu y} dy + \left(t + \frac{1}{\lambda + \mu}\right)e^{-\mu t} \\ &= \frac{1}{\mu} - \left(t + \frac{1}{\mu}\right)e^{-\mu t} + \left(t + \frac{1}{\lambda + \mu}\right)e^{-\mu t} \\ &= \frac{1}{\mu} \left(1 - \frac{\lambda}{\lambda + \mu}e^{-\mu t}\right). \end{aligned}$$

We could also arrive at this result by writing $Y = B + (Y - B)$, and noting that $Y - B$ is the additional amount of service time remaining when the cycle time ends; and thus,

$$\mathbf{E}[B] = \mathbf{E}[Y] + \mathbf{E}[B - Y] = \frac{1}{\mu} - \frac{1}{\mu} \mathbf{P}(Y > B) = \frac{1}{\mu} \left(1 - \frac{\lambda}{\lambda + \mu} e^{-\mu t} \right),$$

which is the same result as above.

In any case, we have that the long-run proportion of time that the server is busy is $\mathbf{E}[B]/(t + \frac{1}{\lambda})$. \square

5. Consider 3 machines, all of which are needed for a system to work. Machine i functions for an $\text{Exp}(\lambda_i)$ time before it fails, $i = 1, 2, 3$. When a machine fails, the system is shut down and repair begins on the failed machine. The time to fix machine 1 is $\text{Exp}(5)$; machine 2 is $\text{Unif}(0,4)$; and machine 3 is $\text{Erlang}_3(2)$. Once a machine is fixed, it's as good as new, and all machines are restarted.

- (a) What proportion of time is the system working?

Solution: This is an alternating renewal (“on/off”) process. We have

$$\begin{aligned} \mathbf{E}[\text{off}] &= \sum_{i=1}^3 \mathbf{E}[\text{off} \mid i \text{ fails}] \mathbf{P}(i \text{ fails}) \\ &= (1/5) \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} + (2) \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} + (3/2) \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}. \end{aligned}$$

Since the on time in a cycle is $\text{Exp}(\lambda_1 + \lambda_2 + \lambda_3)$, we find that the proportion of time that the system is working is

$$p = \frac{1/(\lambda_1 + \lambda_2 + \lambda_3)}{\mathbf{E}[C]},$$

where

$$\mathbf{E}[C] = \mathbf{E}[\text{cycle time}] = 1/(\lambda_1 + \lambda_2 + \lambda_3) + \mathbf{E}[\text{off}]. \quad \square$$

- (b) What proportion of time is machine 1 being repaired?

Solution: Suppose we earn a “reward” of 1 per unit time that machine 1 is being repaired. Then the desired proportion of time is

$$\frac{(1/5) \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}}{\mathbf{E}[C]}. \quad \square$$

- (c) What proportion of time is machine 2 in a state of “suspended animation” (i.e., neither working nor repaired)?

Solution: Suppose we earn a “reward” of 1 per unit time that machine 2 is in suspended animation. Then the desired proportion of time is

$$\frac{(1/5)\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} + (3/2)\frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}}{\mathbb{E}[C]}. \quad \square$$

6. Each time a machine breaks down, it’s replaced by a new one. In the long run, what percentage of time is the machine in use less than one year old if the lifetime distribution of the machine is Unif(0,2)? How about Exp(1)?

Solution: We need to calculate $\frac{1}{\mu} \int_0^1 (1 - F(x)) dx$. For the Unif(0,2) case, this quantity is

$$\frac{1}{\mu} \int_0^1 (1 - F(x)) dx = \int_0^1 (1 - (x/2)) dx = \frac{3}{4}. \quad \square$$

For the Exp(1) case,

$$\frac{1}{\mu} \int_0^1 (1 - F(x)) dx = \int_0^1 e^{-x} dx = 1 - e^{-1}. \quad \square$$

7. Suppose F_e is the c.d.f. of the equilibrium distribution corresponding to the lifetime c.d.f. F , i.e.,

$$F_e(x) = \frac{1}{\mu} \int_0^x (1 - F(y)) dy.$$

- (a) If F is exponential, show that $F_e = F$.

Solution: Assume that $X \sim \text{Exp}(1/\mu)$ (with mean μ). Then

$$\frac{1}{\mu} \int_0^x (1 - F(y)) dy = \frac{1}{\mu} \int_0^x e^{-y/\mu} dy = 1 - e^{-x/\mu}. \quad \square$$

- (b) If the interarrival times are all exactly equal to some constant c , show that the equilibrium distribution is Unif(0, c).

Solution:

$$\frac{1}{\mu} \int_0^x (1 - F(y)) dy = \frac{1}{c} \int_0^x 1 dy = \frac{x}{c}, \quad 0 \leq x \leq c. \quad \square$$

- (c) Suppose that Georgia Tech allows for two hours of parking at all non-metered locations on campus. Parking goons regularly tour around, passing the same point every two hours. When a goon encounters a car, he marks it with chalk. If the same car is there on the goon's return two hours later, then he writes a parking ticket. If you park your car at GT and return after 3 hours, what the probability that you'll receive a ticket?

Solution: You'll get a ticket if, starting when you park, the goon shows up within an hour. Treat the times that the goon shows up as renewals — these are constant renewals every 2 hours. Thus, we want to know if the remaining lifetime of a renewal is less than an hour. By part (b), the equilibrium distribution of the remaining life is $\text{Unif}(0,2)$; and thus, the probability that the goon shows up is $1/2$. \square

8. Consider a renewal process with interarrival c.d.f.

$$F(x) = 1 - \frac{1}{2}e^{-x} - \frac{1}{2}e^{-x/2}, \quad x > 0.$$

In other words, interarrivals are equally likely to be $\text{Exp}(1)$ or $\text{Exp}(1/2)$.

- (a) Without any calculations, guess the equilibrium c.d.f. F_e .

Solution: Since half of the interarrivals are exponential with mean 1 and half are exponential with mean 2, we could guess that the exponentials with mean 2 would last, on average, twice as long. So

$$1 - F_e(x) = \mathbf{P}(X_e > x) = \frac{2}{3}e^{-x/2} + \frac{1}{3}e^{-x}. \quad \square$$

- (b) OK, so prove your guess.

Solution: The mean of this distribution is $\mu = \frac{1}{2}(1) + \frac{1}{2}(2) = \frac{3}{2}$. Then

$$F_e(x) = \frac{1}{\mu} \int_0^x (1 - F(x)) dx = \frac{2}{3} \int_0^x \left[\frac{1}{2}e^{-x} - \frac{1}{2}e^{-x/2} \right] dx.$$

After a little algebra, you indeed get the answer in (a). \square .

9. A system can be in states 1, 2, or 3. Each time the system enters state i , it stays there for a random amount of time having mean μ_i and then makes a transition to state j w.p. P_{ij} . Suppose that $P_{12} = 1$, $P_{21} = P_{23} = 1/2$, and $P_{31} = 1$.

(a) What proportion of transitions takes the system into state 1?

Solution: Using the usual manipulations, you can show that the limiting probabilities for the Markov chain are $\pi_1 = \pi_2 = 2/5$ and $\pi_3 = 1/5$. Thus, the desired answer is $\pi_1 = 2/5$. \square

- (b) If $\mu_1 = 1$, $\mu_2 = 2$, and $\mu_3 = 3$, what proportion of time does the system spend in each state? (Think semi-Markov process.)

Solution: Using the SMP result from class, we have

$$P_i = \frac{\pi_i \mu_i}{\sum_{j=1}^3 \pi_j \mu_j}.$$

This yields $P_1 = 2/9$, $P_2 = 4/9$, and $P_3 = 3/9$. \square

10. In 1984 the country of Morocco tried to determine the average amount of times tourists spent in the country by using two different sampling techniques. In the first one, they questioned randomly chosen tourists as they were leaving the country. In the second one, they questioned randomly chosen guests at hotels. (Each tourist stayed at a hotel.) The average visit time of the 3000 tourists chosen from hotels was 17.8 days, while the average visit time of the 12321 tourists questioned at departure was 9.0 days. Can you explain this discrepancy?

Solution: This is an example of the inspection paradox. Assume that all of the tourists spend i.i.d. amounts of time in the country before departing. Therefore, those surveyed at departure are a good random sample of departing tourists. However, if you question randomly chosen hotel guests, then you actually get a “bi-ased” sample since longer-staying hotel guests are more likely to be questioned! \square

11. Suppose X_i , $i = 1, 2, \dots$ are interarrival times of the RP $N(t)$. Suppose $Y \sim \text{Exp}(\lambda)$ and that everything is independent.

(a) Show that

$$\mathbf{P}\left(\sum_{i=1}^n X_i < Y\right) = [\mathbf{P}(X_1 < Y)]^n.$$

Solution: We have

$$\begin{aligned}
& \mathbb{P}\left(\sum_{i=1}^n X_i < Y\right) \\
&= \mathbb{P}\left(\sum_{i=1}^n X_i < Y \mid X_n < Y\right)\mathbb{P}(X_n < Y) + \mathbb{P}\left(\sum_{i=1}^n X_i < Y \mid X_n > Y\right)\mathbb{P}(X_n > Y) \\
&= \mathbb{P}\left(\sum_{i=1}^n X_i < Y \mid X_n < Y\right)\mathbb{P}(X_n < Y) \\
&= \mathbb{P}\left(\sum_{i=1}^n X_i < Y \mid X_n < Y\right)\mathbb{P}(X_1 < Y). \tag{1}
\end{aligned}$$

Let $Z = \sum_{i=1}^{n-1} X_i$, and let $f_Z(z)$ and $f_X(x)$ denote the p.d.f.'s of Z and X_n , respectively. You can use a similar argument for discrete case.) (There may be a faster way to do this, but...) Conditioning twice, we have

$$\begin{aligned}
& \mathbb{P}\left(\sum_{i=1}^n X_i < Y \mid X_n < Y\right) \\
&= \int_0^\infty \int_0^\infty \mathbb{P}\left(\sum_{i=1}^n X_i < Y \mid X_n < Y, X_n = x, Z = z\right) f_Z(z) f_X(x) dx dz \\
&= \int_0^\infty \int_0^\infty \mathbb{P}(z + x < Y \mid x < Y, X_n = x, Z = z) f_Z(z) f_X(x) dx dz \\
&= \int_0^\infty \int_0^\infty \mathbb{P}(Y > z + x \mid Y > x) f_Z(z) f_X(x) dx dz \quad (\text{by independence}) \\
&= \int_0^\infty \int_0^\infty \mathbb{P}(Y > z) f_Z(z) f_X(x) dx dz \quad (\text{by memoryless}) \\
&= \int_0^\infty \mathbb{P}(Y > z) f_Z(z) dz \\
&= \int_0^\infty \mathbb{P}(Y > z \mid Z = z) f_Z(z) dz \quad (\text{by independence}) \\
&= \int_0^\infty \mathbb{P}(Y > Z \mid Z = z) f_Z(z) dz \\
&= \mathbb{P}(Y > Z).
\end{aligned}$$

Plugging into (1), we have

$$\mathbb{P}\left(\sum_{i=1}^n X_i < Y\right) = \mathbb{P}(Z < Y)\mathbb{P}(X_1 < Y) = \mathbb{P}\left(\sum_{i=1}^{n-1} X_i < Y\right)\mathbb{P}(X_1 < Y).$$

Repeating this argument gives us the desired result. \square

(b) Then show that

$$\mathbf{E}[N(Y)] = \frac{\mathbf{E}[e^{-\lambda X_1}]}{1 - \mathbf{E}[e^{-\lambda X_1}]}.$$

Solution: Using our usual trick for positive random variables and the result of (a), we have

$$\begin{aligned} \mathbf{E}[N(Y)] &= \sum_{n=1}^{\infty} \mathbf{P}(N(Y) \geq n) \\ &= \sum_{n=1}^{\infty} \mathbf{P}\left(\sum_{i=1}^n X_i < Y\right) \\ &= \sum_{n=1}^{\infty} [\mathbf{P}(X_1 < Y)]^n \\ &= \frac{p}{1-p}, \end{aligned}$$

where

$$\begin{aligned} p &= \mathbf{P}(X_1 < Y) \\ &= \int_0^{\infty} \mathbf{P}(X_1 < Y | X_1 = x) f_X(x) dx \\ &= \int_0^{\infty} \mathbf{P}(Y > x) f_X(x) dx \\ &= \int_0^{\infty} e^{-\lambda x} f_X(x) dx \\ &= \mathbf{E}[e^{-\lambda X}]. \quad \square \end{aligned}$$