1. Let $X$ be the outcome of a 4-sided die toss. Find $\text{Var}(\ln(X) + 1)$.

**Solution:** By the law of the Unconscious Statistician,

$$E[\ln(X)] = \sum_{i=1}^{4} \ln(i)P(X = i) = \frac{1}{4} \sum_{i=1}^{4} \ln(i) = 0.7945.$$  
Similarly,  

$$E[\ln^2(X)] = \sum_{i=1}^{4} \ln^2(i)P(X = i) = 0.9023.$$  

Then  

$$\text{Var}(\ln(X) + 1) = \text{Var}(\ln(X)) = E[\ln^2(X)] - (E[\ln(X)])^2 = 0.2711. \quad \diamond$$

2. Suppose that the lifetime of a transistor is exponential with a mean of 10 years. Further suppose that the transistor has already survived 20 years. Find the probability that it will not fail in the next 10 years.

**Solution:** By the memoryless property,

$$P(X \geq 30|X \geq 20) = P(X \geq 10) = e^{-x} = e^{-1} = 0.3679. \quad \diamond$$

3. Suppose that $X \sim c + \text{Exp}(\lambda)$, where the time units are in years. $X$ could represent the lifetime of a lightbulb that is *guaranteed* to last at least $c$ years. Find the median of $X$, that is, the point $m$ such that $P(X \leq m) = P(X > m) = 0.5$.

**Solution:** Let $Y \sim \text{Exp}(\lambda)$. Then

$$0.5 = P(X \leq m) = P(c + Y \leq m) = P(Y \leq m - c) = 1 - e^{-\lambda(m-c)}.$$
This is true iff 

\[-\lambda (m-c) = \ell n(0.5) \quad \text{iff} \quad m = c + \frac{\ell n(2)}{\lambda}. \]

4. Suppose \(X_1, X_2,\) and \(X_3\) are i.i.d. Bernoulli\(p\) random variables, which represent the functionality of three network components. Think of a signal passing through a network, where \(X_i = 1\) if the signal can successfully get through component \(i\), for \(i = 1, 2, 3\) (and \(X_i = 0\) if the signal is unsuccessful). Let’s consider two set-ups:

(*) \(X_1, X_2, X_3\) have \(p = 0.95\) and are hooked up in series so that a signal getting through the network has to pass through all components 1 AND 2 AND 3.

(**) \(X_1, X_2, X_3\) have \(p = 0.8\) and are hooked up in parallel so that a signal getting through the network only has to pass through components 1 OR 2 OR 3.

Which series is more reliable, i.e., more likely to permit a signal to pass through — (*) or (**)?

Solution:

\[
P(*) = P\left( \bigcap_{i=1}^{3} (X_i = 1) \right) = \prod_{i=1}^{3} P(X_i = 1) = 0.8574. \]

Meanwhile, by inclusion-exclusion and the fact that the \(X_i\)’s are i.i.d., we have

\[
P(**) = P\left( \bigcup_{i=1}^{3} (X_i = 1) \right) = P(X_1 = 1) + P(X_2 = 1) + P(X_3 = 1) - P(X_1 = 1 \cap X_2 = 1) - P(X_1 = 1 \cap X_3 = 1) - P(X_2 = 1 \cap X_3 = 1) + P(X_1 = 1 \cap X_2 = 1 \cap X_3 = 1)
= 3P(X_1 = 1) - 3P^2(X_1 = 1) + P^3(X_1 = 1)
= 3p - 3p^2 + p^3 = 0.992.
\]

Or, now that I think of it, we could’ve gotten this result way more quickly by

\[
P(**) = P\left( \bigcup_{i=1}^{3} (X_i = 1) \right) = 1 = P\left( \bigcap_{i=1}^{3} (X_i = 0) \right) = 1 - (1-p)^3 = 0.992.
\]

In any case, (**) is more reliable. \(\Diamond\)
5. TRUE or FALSE? If $X$ is any normal distribution, then about 95% of all observations from $X$ will fall within two standard deviations of the mean.

**Solution:** TRUE.

6. TRUE or FALSE? The normal quantile value $\Phi^{-1}(0.975) = 1.96$.

**Solution:** TRUE.

7. Suppose $X \sim \text{Nor}(1, 1)$. Find $c$ such that $P(-c \leq X \leq c) = 0.95$. Careful — this may require a little trial-and-error.

**Solution:** Standardizing with $Z \sim \text{Nor}(0, 1)$, we have

$$0.95 = P(-c \leq X \leq c) = P(-c - 1 \leq Z \leq c - 1) = \Phi(c - 1) - \Phi(-c - 1).$$

Now we have to find the value of $c$ that satisfies this equation. I’ll go to Excel and use the `NORM.S.DIST` function to evaluate $\Phi(c - 1) - \Phi(-c - 1)$ for various values of $c$. After a little trial and error, I find that $c = 2.65$ pretty much does the trick.

8. Suppose that $X$ and $Y$ are the scores that a Georgia Tech student and his twin who goes to the Univ. of Georgia will receive, respectively, on the same math test. Further suppose that $X \sim \text{Nor}(90, 100)$, $Y \sim \text{Nor}(60, 100)$ and $\text{Cov}(X,Y) = 50$. Find the probability that the GT kid will beat the UGA kid by at least 40 points. (You can assume that $X - Y$ is normal.)

**Solution:** Note that $E[X - Y] = 30$ and

$$\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X,Y) = 100.$$ 

Therefore, $X - Y \sim N(30, 100)$. This implies that

$$P(X - Y > 40) = P \left( Z > \frac{40 - 30}{\sqrt{100}} \right) = P(Z > 1) = 0.1587.$$
9. If \( X_1, \ldots, X_{400} \) are i.i.d. from some distribution with mean 0 and variance 1600, find the approximate probability that the sample mean \( \bar{X} \) is between -2 and 2.

**Solution:** By the Central Limit Theorem, \( \bar{X} \) will be approximately normal. Let’s find its mean and variance. To this end,

\[
\mathbb{E}[\bar{X}] = \mathbb{E}[X_i] = 0
\]

and

\[
\text{Var}(\bar{X}) = \frac{\text{Var}(X_i)}{n} = \frac{1600}{400} = 4.
\]

Thus, \( \bar{X} \approx \text{Nor}(0, 4) \), and so

\[
P(-2 \leq \bar{X} \leq 2) \approx P(-1 \leq Z \leq 1) = 2\Phi(1) - 1 = 2(0.8413) - 1 = 0.6826.
\]

10. **TRUE or FALSE?** Consider any i.i.d. sequence of random variables having finite variance. Then the Central Limit Theorem says that a properly standardized sample mean can be approximated by a standard normal random variable as the sample size becomes large.

**Solution:** TRUE. ♦

11. **Short Research Question:** Go to The Internets and write up a couple of paragraphs on who “invented” the Central Limit Theorem (there are several reasonably correct answers).

12. If \( X \sim \chi^2(5) \), find \( P(X < 11.07) \).

**Solution:** 0.95. ♦

13. **TRUE or FALSE?** \( t_{0.025, 8} > z_{0.025} \).

**Solution:** TRUE, since \( t_{0.025, 8} = 2.306 > 1.96 = z_{0.025} \). In fact, since the \( t \)-distribution has fatter tails than the standard normal, you don’t even need tables for this one. ♦
14. Suppose \( T \sim t(343) \). What’s \( P(T < 1) \)?

**Solution:** Because of the high degrees of freedom, \( P(T < 1) \approx P(Z < 1) = 0.8413 \).

15. TRUE or FALSE? \( P(F(5, 3) < F_{0.95,5,3}) = P(F(5, 3) < 1/F_{0.05,5,3}) \).

**Solution:** FALSE, since \( F_{0.95,5,3} = 1/F_{0.05,5,3} \) (flip the d.f.).

16. Suppose \( X_1, \ldots, X_6 \sim \text{Nor}(3, 9) \), \( Y_1, \ldots, Y_7 \sim \text{Nor}(-3, 2) \), and everything is independent. Let \( S_X^2 \) and \( S_Y^2 \) denote the sample variances of the \( X_i \)'s and \( Y_j \)'s, respectively. Name the distribution (with parameter(s)) of \( S_Y^2 / S_X^2 \).

**Solution:** \( S_X^2 \sim \sigma_X^2 \chi^2(5) / 5 \) and \( S_Y^2 \sim \sigma_Y^2 \chi^2(6) / 6 \), where \( \sigma_X^2 = \text{Var}(X_i) = 9 \) and \( \sigma_Y^2 = \text{Var}(Y_j) = 2 \). Therefore,

\[
\frac{S_Y^2}{S_X^2} \sim \frac{2\chi^2(6)/6}{9\chi^2(5)/5} \sim \frac{2}{9} F(6, 5).
\]

17. Suppose \( X_1, \ldots, X_n \) are i.i.d. \( \text{Exp}(\lambda) \).

(a) TRUE or FALSE? The sample mean \( \bar{X} \) is unbiased for the mean \( 1/\lambda \).

**Solution:** TRUE. (The sample mean is always unbiased for the true mean.)

(b) TRUE or FALSE? \( 1/\bar{X} \) is unbiased for \( \lambda \).

**Solution:** FALSE. (It’s almost unbiased, but not quite; in fact, see the next question.)

(c) Find \( E[1/\bar{X}] \). (This might take a little work if you do it from scratch.)

**Solution:** Let’s first look at \( Y = \sum_{i=1}^n X_i \) (so that \( \bar{X} = Y/n \)). Since \( Y \) is the sum of i.i.d. \( \text{Exp}(\lambda) \) random variables, we know that \( Y \sim \text{Erlang}_n(\lambda) \), and so
$Y$ has p.d.f.

$$f_Y(y) = \frac{\lambda^n y^{n-1} e^{-\lambda y}}{(n-1)!}, \quad y > 0,$$

Then by LOTUS,

$$E[1/\bar{X}] = nE[1/Y]$$

$$= n \int_0^\infty \frac{1}{y} \frac{\lambda^n y^{n-1} e^{-\lambda y}}{(n-1)!} dy$$

$$= \frac{n\lambda}{n-1} \int_0^\infty \frac{\lambda^{n-1} y^{n-2} e^{-\lambda y}}{(n-2)!} dy$$

$$= \frac{\lambda n}{n-1},$$

since the thing inside the integral is the p.d.f. of the $\text{Erlang}_{n-1}(\lambda)$ distribution.

(d) TRUE or FALSE? $1/\bar{X}$ is the MLE for $\lambda$.

**Solution:** TRUE. ($\bar{X}$ is the MLE for the mean $1/\lambda$, so the result follows by invariance of MLE’s.)

18. Suppose $X_1, X_2, X_3$ are i.i.d. $\text{Nor}(\mu, \sigma^2)$, and we observe $X_1 = 7$, $X_2 = 1$, and $X_3 = 4$.

(a) What is the sample variance of the $X_i$’s?

**Solution:**

$$S^2 = \frac{\sum_{i=1}^n X_i^2 - n\bar{X}^2}{n-1} = 9.$$  

(b) What is the maximum likelihood estimate of $\sigma^2$?

**Solution:** $\bar{X} = 4$ and $n = 3$. Thus,

$$\hat{\sigma}^2 = \frac{n-1}{n} S^2 = 6.$$
(c) What is the maximum likelihood estimate of $P(X_i > 5)$?

**Solution:** Since

$$P(X_i > 5) = P\left(Z > \frac{5 - \mu}{\sigma}\right) = 1 - \Phi\left(\frac{5 - \mu}{\sigma}\right) = \Phi\left(\frac{\mu - 5}{\sigma}\right),$$

invariance gives us

$$\hat{P}(X_i > 5) = \Phi\left(\frac{\hat{\mu} - 5}{\hat{\sigma}}\right) = \Phi\left(\frac{\bar{X} - 5}{\sqrt{\frac{(n-1)S^2}{n}}}\right) = \Phi\left(\frac{4 - 5}{\sqrt{6}}\right) = 0.3415. \diamond$$

19. Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. Geom($p$). Thus, for all $i$, we have $P(X_i = k) = (1 - p)^{k-1}p$, for $k = 1, 2, \ldots, n$. What is the maximum likelihood estimate of $p$?

**Solution:** $\hat{p} = 1/\bar{X}$. \diamond

In case you missed it, here’s the proof. The likelihood function is

$$L(p) = \prod_{i=1}^{n} P(X_i = x_i) = (1 - p)^{\sum_{i=1}^{n} x_i - np}$$

so that

$$\ell_n(L(p)) = \left(\sum_{i=1}^{n} x_i - n\right) \ell_n(1 - p) + n \ell_n(p).$$

Thus, setting

$$\frac{d}{dp} \ell_n(L(p)) = -\frac{\left(\sum_{i=1}^{n} x_i - n\right)}{1 - p} + \frac{n}{p} = 0,$$

we have (after a little algebra) $\hat{p} = 1/\bar{X}$, as promised. \diamond

20. Suppose $X_1, X_2, X_3$ are i.i.d. Unif($\theta, 0$), and we observe $X_1 = -7$, $X_2 = -1$, and $X_3 = -4$. What is the maximum likelihood estimate of $\theta$?

**Solution:** By symmetry from an example in class, we have $\hat{\theta} = \min_i X_i = -7$. \diamond
21. Suppose $X_1, X_2, X_3$ are i.i.d. $\text{Nor}(\mu, 16)$. Define two estimators for $\mu$: $T_1 \equiv X_1 - X_2 + X_3$ and $T_2 \equiv (4X_1 + 3X_2 + X_3)/8$. Which of $T_1$ or $T_2$ has the smaller MSE?

**Solution:** It is easy to show that $E[T_1] = E[T_2] = \mu$, so that both estimators are unbiased for $\mu$. Thus, in this case, $\text{MSE}(T_1) = \text{Var}(T_1)$ and $\text{MSE}(T_2) = \text{Var}(T_2)$.

First, $\text{Var}(T_1) = \text{Var}(X_1 - X_2 + X_3) = \text{Var}(X_1) + \text{Var}(X_2) + \text{Var}(X_3) = 48$.

Similarly, 

$$\text{Var}(T_2) = \text{Var}\left(\frac{4X_1 + 3X_2 + X_3}{8}\right) = \frac{16\text{Var}(X_1) + 9\text{Var}(X_2) + \text{Var}(X_3)}{64} = 6.5.$$ 

Therefore, $T_2$ has lower MSE. ♦.

22. Which family member is actually an estimation method?

(a) CAT  
(b) DAD  
(c) MOM  
(d) BRO  
(e) SIS

**Solution:** (c) MOM (method of moments). ♦

23. Suppose $X_1, \ldots, X_{10}$ are i.i.d. normal with unknown mean and *known* variance $\sigma^2 = 49$. Further suppose that $\bar{X} = -50$ and $S^2 = 60$.

(a) Find a 99% two-sided confidence interval for $\mu$.

**Solution:** You can ignore the red herring information about $S^2$ (since we know that $\sigma^2 = 49$).

$$\mu \in \bar{X} \pm z_{a/2} \sqrt{\frac{\sigma^2}{n}} = -50 \pm 2.576 \sqrt{\frac{49}{10}} = -50 \pm 5.70 = [-55.70, -44.30].$$ ♦
(b) Find a 99% two-sided confidence interval for $2\mu - 4$.

**Solution:** For any confidence interval for the mean $\mu$ with lower and upper bounds $L$ and $U$, we have

$$1 - \alpha = P(L \leq \mu \leq U) = P(2L - 4 \leq 2\mu - 4 \leq 2U - 4).$$

Using the $L$ and $U$ bounds from Question 23a, we have

$$2\mu - 4 \in [-115.40, -92.60]. \diamond$$

24. Suppose $X_1, \ldots, X_{10}$ are i.i.d. normal with unknown mean and unknown variance $\sigma^2$. Further suppose that $\bar{X} = -50$ and $S^2 = 60$.

(a) Find a 99% two-sided confidence interval for $\mu$.

**Solution:** Since $t_{0.005,9} = 3.250$, we have

$$\mu \in \bar{X} \pm t_{0.005,9-1} \sqrt{S^2/n} = -50 \pm 3.250 \sqrt{60/10} = -50 \pm 7.96 = [-57.96, -42.04]. \diamond$$

(b) Is your answer to Question 24a wider or narrower than your answer to Question 23a? Why?

**Solution:** They’re wider. First of all, the sample variance $S^2$ happens to be larger than the actual variance $\sigma^2$ (by bad luck). Second, we pay a penalty for not knowing the variance by having to use the $t$ quantile instead of the $z$ quantile. \diamond

(c) Find a 99% two-sided confidence interval for $\sigma^2$.

**Solution:**

$$\sigma^2 \in \left[ \frac{(n-1)S^2}{\chi^2_{0.005, n-1}}, \frac{(n-1)S^2}{\chi^2_{0.995, n-1}} \right] = \left[ \frac{540}{23.59}, \frac{540}{1.735} \right] = [22.89, 311.24]. \diamond$$
25. Suppose that $X_1, \ldots, X_n$ are i.i.d. Bernoulli with unknown mean $p$, and that we have carried out a preliminary investigation suggesting $p \approx 0.9$. How big would $n$ have to be in order for a two-sided 99% confidence interval to have a half-length of 0.01? (Give the smallest such number.)

**Solution:**

$$z_{\alpha/2} \frac{\hat{p}(1 - \hat{p})}{n} \leq \varepsilon^2.$$ So,

$$n \geq z_{\alpha/2}^2 \frac{\hat{p}(1 - \hat{p})}{\varepsilon^2} = \frac{2.576^2 \cdot 0.9 \cdot 0.1}{(0.01)^2} = 5973 \text{ (rounded up).}$$

26. After collecting your set of observations, what happens to the length of a confidence interval for the mean as the confidence level moves from 90% to 95%?

(a) It increases.
(b) It decreases.
(c) It stays the same.
(d) You can’t tell.

**Solution:** (a) It increases.

27. Consider i.i.d. normal observations $X_1, \ldots, X_{10}$ with unknown mean $\mu$ and unknown variance $\sigma^2$. What is the expected width of the usual 95% two-sided confidence interval for $\sigma^2$? You can keep your answer in terms of $\sigma$.

**Solution:** The confidence interval is of the form

$$\sigma^2 \in \left[ \frac{(n - 1)S^2}{\chi^2_{\frac{\alpha}{2}, n-1}}, \frac{(n - 1)S^2}{\chi^2_{1 - \frac{\alpha}{2}, n-1}} \right].$$

Thus, the length is

$$L = \frac{(n - 1)S^2}{\chi^2_{1 - \frac{\alpha}{2}, n-1}} - \frac{(n - 1)S^2}{\chi^2_{\frac{\alpha}{2}, n-1}},$$

and so the expected length is

$$E[L] = (n - 1) \left[ \frac{1}{\chi^2_{0.975,9}} - \frac{1}{\chi^2_{0.025,9}} \right] \cdot E[S^2] = 9 \left[ \frac{1}{2.700} - \frac{1}{19.02} \right] \sigma^2 = 2.86\sigma^2.$$
28. Suppose we conduct an experiment to test to see if people can throw farther right- or left-handed. We get 20 people to do the experiment. Each throws a ball right- handed once and throws a ball left-handed once, and we measure the distances. If we are interested in determining a confidence interval for the mean difference in left- and right-handed throws, which type of c.i. would we likely use?

(a) $z$ (normal) confidence interval for differences
(b) pooled $t$ confidence interval for differences
(c) paired $t$ confidence interval for differences
(d) $\chi^2$ confidence interval for differences
(e) $F$ confidence interval for differences

Solution: (c)  

29. TRUE or FALSE? We reject the null hypothesis if we are given statistically significant evidence that it is false.

Solution: TRUE.  

30. The quantity $\alpha$ is known as

(a) $P$(Type I error)
(b) $P$(Type II error)
(c) level of significance
(d) $P$(Reject $H_0$ | $H_0$ is true)

Solution: (a), (c), and (d).  

31. Suppose that we examine the IQs of 50 Justin Bieber concert attendees. We assume that the IQs are normally distributed with a standard deviation of 10. Suppose that the sample mean turns out to be 82. Test the null hypothesis that the mean IQ of the attendees is at least 90. Use $\alpha = 0.05$.

Solution: The null hypothesis is $H_0 : \mu \geq 90$. The test statistic is

$$Z_0 = \frac{\bar{X} - \mu_0}{\sqrt{\sigma^2/n}} = \frac{82 - 90}{\sqrt{100/50}} = -5.66.$$
Since \( Z_0 < -z_{\alpha} = -1.645 \), we reject \( H_0 \). (We didn’t even really need tables for this, since it’s so obvious.)

32. Referring to Question 31, how many observations should we take if we want the probability of a Type II error to be 0.10 when \( \mu \) happens to equal 87?

**Solution:** Since this is a one-sided test,

\[
 n \approx \frac{\sigma^2(z_{\alpha} + z_{\beta})^2}{\delta^2} = \frac{\sigma^2(z_{0.05} + z_{0.10})^2}{(87 - 90)^2} = \frac{100(1.645 + 1.28)^2}{9} = 95.1 = 96.
\]

33. Suppose we want to compare the means of two normal populations, both of which have *unknown but approximately equal variances*. We take \( n = 6 \) observations from the first population and find that the sample mean and sample variance are \( \bar{x} = 50 \) and \( s_x^2 = 120 \). We take \( m = 5 \) observations from the second population and find that the sample mean and sample variance are \( \bar{y} = 75 \) and \( s_y^2 = 100 \). Test the hypothesis that \( \mu_x = \mu_y \) with \( \alpha = 0.05 \), i.e., either accept or reject.

**Solution:** First of all, the pooled variance is

\[
 S_p^2 = \frac{(n - 1)S_x^2 + (m - 1)S_y^2}{n + m - 2} = 11.11.
\]

Then the CI is

\[
 \mu_x - \mu_y \in \bar{X} - \bar{Y} \pm t_{\alpha/2,n+m-2} \sqrt{S_p^2\left(\frac{1}{n} + \frac{1}{m}\right)}
\]

\[
 = -25 \pm t_{0.025,9} \sqrt{11.11\left(\frac{1}{6} + \frac{1}{5}\right)}
\]

\[
 = -25 \pm 2.262(6.383)
\]

\[
 = -25 \pm 14.44.
\]

Since the CI doesn’t contain 0, we *reject* \( H_0 : \mu_x = \mu_y \).

34. Short Research Question: Go to The Internets and write up a couple of paragraphs on what a “goodness-of-fit” hypothesis test is.

35. Let’s see if weight depends on height. We consider 6 i.i.d. people.
(a) Fit a regression line to this data and report your estimates for $\beta_0$ and $\beta_1$.

**Solution:** After the usual algebra (which I did in Minitab because I’m lazy), we obtain the model $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x = -74.5 + 3.66x$. ♦

(b) What is the expected weight of a person who is 72 inches tall?

**Solution:** $\hat{y}|x = -74.5 + 3.66(72) = 188.7$. ♦

(c) Give a 95% confidence interval for $\beta_1$.

**Solution:** First of all, note that $\bar{x} = 67.17$ and $S_{xx} = \sum_{i=1}^{n} x_i^2 - nx^2 = 148.83$. Moreover,

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{n - 2} = \frac{\sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}{n - 2} = 182.8.$$

The desired confidence interval is therefore

$$\beta_1 \in \hat{\beta}_1 \pm t_{\alpha/2, n-2} \frac{\hat{\sigma}}{\sqrt{S_{xx}}} = \hat{\beta}_1 \pm t_{0.025, 4} \frac{\hat{\sigma}}{\sqrt{S_{xx}}} = 3.66 \pm (2.776)(1.11) = 3.66 \pm 3.08. \quad ♦$$