This is a take-home test. But please limit the total work time to less than about 3 hours. If you have a question, please send me an email or give me a call.

1. Suppose that $X$ has moment generating function $M_X(t) = \frac{3e^{2t}}{3-t}$ for $t < 3$.

   (a) Find $E[X]$.

   **Solution:** First let’s do it the most straightforward way.

   $\mathbb{E}[X] = \left. \frac{d}{dt} M_X(t) \right|_{t=0}$
   
   $= \frac{d}{dt} \frac{3e^{2t}}{3-t} \bigg|_{t=0}$
   
   $= \frac{(3-t)6e^{2t} + 3e^{2t}}{(3-t)^2} \bigg|_{t=0}$
   
   $= \frac{21e^{2t} - 6te^{2t}}{(3-t)^2} \bigg|_{t=0}$
   
   $= 7/3. \quad \Box$

   Here’s another way to do the problem, which involves identifying the underlying distribution as a linear function of an exponential random variable. Namely, note that $M_X(t) = e^{2t} M_Y(t)$, where $Y \sim \text{Exp}(3)$, thus implying (from a HW problem, that $X = Y + 2$, so that $E[X] = \frac{1}{3} + 2 = 7/3. \quad \Box$

   (b) Find $\text{Var}(X)$. (Be patient... this gets a little tedious!)

   **Solution:** Of course, you can use the straightforward way and take the second derivative to get $E[X^2]$ and then $\text{Var}(X)$, but I’ll be lazy here and use the fact that $X = Y + 2$ to obtain $\text{Var}(X) = \text{Var}(Y) = 1/9. \quad \Box$
2. Suppose $X$ and $Y$ are discrete random variables with the following joint p.m.f., where any letters denote probabilities that you might need to figure out.

<table>
<thead>
<tr>
<th>$f(x,y)$</th>
<th>$X = -3$</th>
<th>$X = 0$</th>
<th>$X = 5$</th>
<th>$P(Y = y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y = 1.6$</td>
<td>$a$</td>
<td>$c$</td>
<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>$Y = 27$</td>
<td>$b$</td>
<td>0</td>
<td>0.3</td>
<td>$d$</td>
</tr>
<tr>
<td>$P(X = x)$</td>
<td>$e$</td>
<td>0.2</td>
<td>$f$</td>
<td>$g$</td>
</tr>
</tbody>
</table>

(a) Fill in the table for $a$, $b$, . . . .

**Solution:** Easy to show that we have

<table>
<thead>
<tr>
<th>$f(x,y)$</th>
<th>$X = -3$</th>
<th>$X = 0$</th>
<th>$X = 5$</th>
<th>$P(Y = y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y = 1.6$</td>
<td>0</td>
<td>0.2</td>
<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>$Y = 27$</td>
<td>0.4</td>
<td>0</td>
<td>0.3</td>
<td>0.7</td>
</tr>
<tr>
<td>$P(X = x)$</td>
<td>0.4</td>
<td>0.2</td>
<td>0.4</td>
<td>1.0</td>
</tr>
</tbody>
</table>

(b) Find $P(X \leq 0)$.

**Solution:** Then $P(X \leq 0) = P(X = -3) + P(X = 0) = 0.6$. □

(c) Find $E[X]$.

**Solution:** $E[X] = \sum_x xP(X = x) = (-3)(0.4) + 0(0.2) + 5(0.4) = 0.8$. □

(d) Find $\text{Cov}(X,Y)$.

**Solution:** Similarly, we have

$$E[Y] = \sum_y yP(Y = y) = 19.38$$

and

$$E[XY] = \sum_x \sum_y xyf(x,y) = 8.9,$$

so that

(e) Are $X$ and $Y$ independent?

**Solution:** No (since $\text{Cov} \neq 0$). □

3. Suppose that $f(x,y) = cx$, for $0 \leq y \leq x \leq 2$.

(a) Find $c$.

**Solution:** $1 = \int_0^2 \int_0^x c \, dy \, dx = 8c/3$, so that $c = 3/8$. □

(b) Find $P(X > 1$ and $Y < 1/2)$.

**Solution:** $\int_1^2 \int_0^{1/2} \frac{3x}{8} \, dy \, dx = 9/32$. □

(c) Find the marginal p.d.f. of $X$.

**Solution:** $f_X(x) = \int_0^x \frac{3x}{8} \, dy = \frac{3x^2}{8}, \ 0 \leq x \leq 2$. □

(d) Find the conditional p.d.f. of $Y$ given that $X = x$.

**Solution:** $f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{x}, \ 0 \leq y \leq x$. □

(e) Find $E[Y|X = x]$.

**Solution:** $E[Y|X] = \int_0^x y f(y|x) \, dy = \frac{x}{2}, \ 0 \leq x \leq 2$. □

(f) Find $E[E[Y|X]]$.

**Solution:** $\int_0^2 E[Y|x]f_X(x) \, dx = \int_0^2 \frac{x}{2} \frac{3x^2}{8} \, dx = 3/4$. □

By the way, note that

$$f_Y(y) = \int_y^2 \frac{3x}{8} \, dx = \frac{3(4-y^2)}{16}, \ 0 \leq y \leq 2.$$
This implies that
\[ E[Y] = \int_0^2 y \frac{3(4 - y^2)}{16} \, dy = \frac{3}{4}, \]
which makes sense since \( E[E[Y|X]] = E[Y] \).

(g) Find \( \text{Cov}(X, Y) \).

\textbf{Solution:} First of all,
\[ E[X] = \int_0^2 x \frac{3x^2}{8} \, dx = \frac{3}{2}, \]
and
\[ E[XY] = \int_0^2 \int_0^x \frac{3x^2y}{8} \, dy \, dx = \frac{6}{5}. \]
This implies that \( \text{Cov}(X, Y) = E[XY] - E[X]E[Y] = \frac{3}{40}. \)

(h) Are \( X \) and \( Y \) independent?

\textbf{Solution:} No. \n
4. Let’s play Name That Distribution! In each question, name the distribution (with parameters, if appropriate).

(a) I am the sum of 15 i.i.d. Bern(0.3) random variables.

\textbf{Solution:} Bin(15, 0.3). \n
(b) I am the only discrete distribution with the memoryless property.

\textbf{Solution:} Geometric. \n
(c) I am the only continuous distribution with the memoryless property.

\textbf{Solution:} Exponential. \n
(d) I am the limiting distribution of $\sqrt{n}(\bar{X} - \mu)/\sigma$ as $n \to \infty$.

**Solution:** Nor(0,1). □

(e) I am the time between two arrivals from a Poisson process with rate $1/(3 \text{ hr})$.

**Solution:** Exp(1/3). □

(f) I have m.g.f. $\exp(3t + 4t^2)$.

**Solution:** The Nor($\mu, \sigma^2$) m.g.f. is $M_X(t) = e^{\mu t + \sigma^2 t^2/2}$, so the answer here is Nor(3, 8). □

(g) I can be represented as $F(X)$, where $X$ is a Normal($-2, 225$) random variable and $F(x)$ is its p.d.f.

**Solution:** Unif(0,1) (by the Inverse Transform Theorem). □

5. Buses show up at the bus stop randomly according to a Poisson process with a rate of 3 per hour. Let’s suppose that I arrive at the stop at 4:00PM; but, sadly, I’m still waiting for a bus at 5:00PM. What’s the probability that a bus will finally arrive within the next 30 minutes?

**Solution:** Let $X$ denote the waiting time. Then by the memoryless property,

$$P(X \leq 90|X > 60) = 1 - P(X > 90|X > 60)$$
$$= 1 - P(X > 30)$$
$$= 1 - e^{-3/2} = 0.777.$$ □

6. Suppose that I buy a package of 3 lightbulbs, and that their lifetimes are i.i.d. Exponential with a mean of 6 months. When a bulb fails, I immediately replace it with the next one from the package. What’s the probability that the total lifetime of the bulbs will be at least a year?
Solution: The total lifetime $X$ is Erlang$_3(2/\text{year})$. Thus,

$$P(X > 1) = \sum_{i=0}^{k-1} e^{-\lambda t} \frac{(\lambda t)^i}{i!} = \sum_{i=0}^{2} e^{-2} \frac{2^i}{i!} = 5e^{-2} = 0.677. \quad \Box$$

7. (This is a bit of an open-ended problem.) The failure rate of a positive random variable $X$ can be regarded as the instantaneous rate of death — that is, the rate of death, given that the person (or lightbulb) has survived until time $x$. It’s formally defined as $f(x)/(1 - F(x))$, where $f(x)$ and $F(x)$ are the p.d.f. and c.d.f. of $X$. Meanwhile, you may recall that the Weibull($a, b$) distribution has c.d.f.

$$F(x) = 1 - \exp[-(ax)^b], \quad x > 0.$$ 

Study the failure rate of $X \sim \text{Weibull}(a, b)$ for various choices of $a$ and $b$.

Solution: Various solutions. You will find that for some choices of $a$ and $b$, the failure rate is monotone increasing, for some it’s monotone decreasing, and for $b = 1$ (the exponential case) it’s flat. \quad \Box

8. Median Questions.

(a) TRUE or FALSE? The mean of any normal distribution is the same as its median. [Note that the median of a continuous RV $X$ is simply the point $x$ such that $P(X \leq x) = P(X \geq x) = 0.5$.]

Solution: TRUE (by symmetry). \quad \Box

(b) TRUE or FALSE? The mean of the Exp($\lambda$) is greater than its median.

Solution: By definition, we can get the median $\tilde{m}$ by solving

$$0.5 = F(\tilde{m}) = 1 - e^{-\lambda \tilde{m}},$$

so that

$$\tilde{m} = \frac{\ln(2)}{\lambda} = \frac{0.693}{\lambda} < \frac{1}{\lambda} = E[X].$$

So the answer is TRUE. \quad \Box
9. Suppose \( X \sim \text{Nor}(1, 9) \). Find the point \( x \) such that \( P(-x \leq X \leq x) = 0.95 \).
(Careful — this isn’t quite symmetric.)

**Solution:** We have

\[
0.95 = P(-x \leq X \leq x) = P\left(-\frac{x-1}{3} \leq Z \leq \frac{x-1}{3}\right) = \Phi\left(\frac{x-1}{3}\right) - \Phi\left(-\frac{x-1}{3}\right). 
\]

Now go to the tables, and find by trial-and-error (or via bisection) the value of \( x \) that approximately solves the equation. You’ll get \( x \approx 6.1923 \). □

10. Suppose that a random man’s height from a certain population is \( \text{Nor}(70, 9) \) and a random woman’s height is \( \text{Nor}(66, 16) \). Find the probability that a random woman is shorter than a random man.

**Solution:** Note that \( W - M \sim \text{Nor}(-4, 25) \). Then we have

\[
P(W < M) = P(W - M < 0) = P\left(Z < \frac{0 - (-4)}{\sqrt{25}}\right) = \Phi(0.8) = 0.788. \quad \square
\]

11. Suppose that \( X_1, X_2, \ldots, X_n \) are i.i.d. \( \text{Nor}(\mu, 25) \), and let \( \bar{X} \) denote the sample mean of these \( n \) observations. What’s the smallest value of \( n \) that will guarantee that \( P\left(|\bar{X} - \mu| \leq 0.5\right) \geq 0.995?\)

**Solution:** We want

\[
0.995 \leq P\left(|\bar{X} - \mu| \leq 0.5\right) \\
= P\left(-0.5 \leq \bar{X} - \mu \leq 0.5\right) \\
= P\left(-0.5 \sqrt{25/n} \leq Z \leq 0.5 \sqrt{25/n}\right) \\
= P\left(-0.1 \sqrt{n} \leq Z \leq 0.1 \sqrt{n}\right) \\
= 2\Phi(0.1 \sqrt{n}) - 1.
\]

This occurs if and only if \( \Phi(0.1 \sqrt{n}) \geq 0.995 \), so that we need

\[
n \geq 100\left(\Phi^{-1}(0.9975)\right)^2 = 100(2.807)^2 = 787.9 \approx 788. \quad \square
\]
12. There are exactly 100 cookies in a bag of Biebler chocolate chip cookies. Let $X_i$ denote the number chocolate chips in cookie $i$. If $X_1, \ldots, X_{100}$ are i.i.d. from a Pois(9) distribution, find the approximate probability that the total number of chocolate chips in the entire bag is at least 925.

**Solution:** By the CLT,

$$\sum_{i=1}^{100} X_i \approx \text{Nor}(n\mu, n\sigma^2) \sim \text{Nor}(n\lambda, n\lambda) \sim \text{Nor}(900, 900).$$

Thus,

$$P\left(\sum_{i=1}^{100} X_i > 925\right) \approx P\left(Z > \frac{925 - 900}{\sqrt{900}}\right) = 1 - \Phi(5/6) = 0.202. \quad \square$$

13. What is your favorite memory about Justin Bieber?